Control of Industrial Robots

Prof. Rocco

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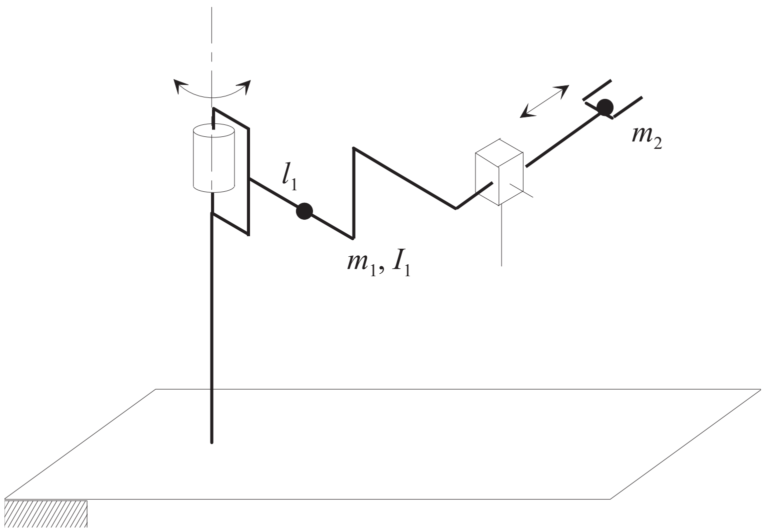
SOLUTION

Control of Industrial Robots
Prof. Paolo Rocco

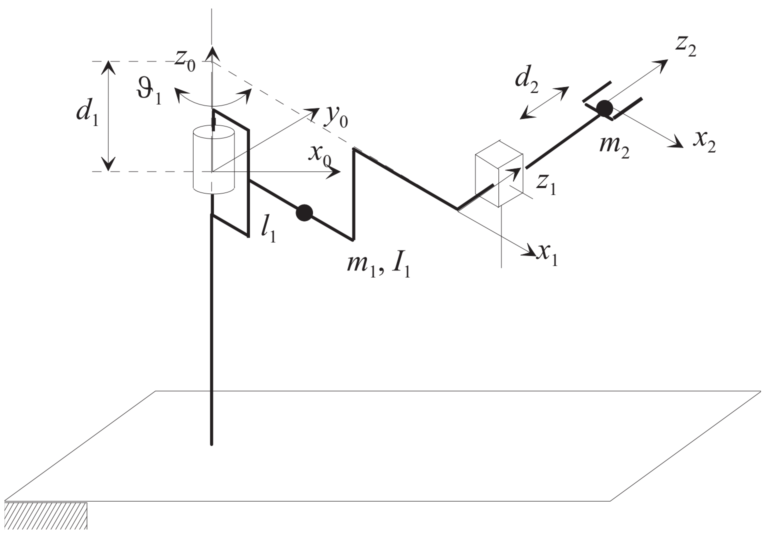
EXERCISE 1

1. Consider the manipulator sketched in the picture, where the mass of the second link is assumed to be concentrated at the end-effector:



Find the expression of the inertia matrix $\mathbf{B}(\mathbf{q})$ of the manipulator.

Denavit-Hartenberg frames can be defined as sketched in this picture:



Computations of the Jacobians:

Link 1

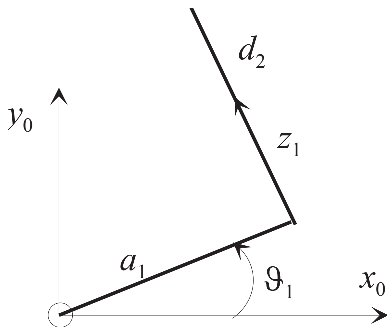
$$\mathbf{J}_P^{(l_1)} = \left[\begin{array}{cc} \mathbf{j}_{P_1}^{(l_1)} & \mathbf{0} \end{array} \right] = \left[\begin{array}{cc} \mathbf{z}_0 \times (\mathbf{p}_{l_1} - \mathbf{p}_0) & \mathbf{0} \end{array} \right] = \left[\begin{array}{cc} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{array} \right]$$

$$\mathbf{J}_O^{(l_1)} = \left[\begin{array}{cc} \mathbf{j}_{O_1}^{(l_1)} & \mathbf{0} \end{array} \right] = \left[\begin{array}{cc} \mathbf{z}_0 & \mathbf{0} \end{array} \right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{array} \right]$$

Link 2

$$\mathbf{J}_P^{(l_2)} = \left[\begin{array}{cc} \mathbf{j}_{P_1}^{(l_2)} & \mathbf{j}_{P_2}^{(l_2)} \end{array} \right] = \left[\begin{array}{cc} \mathbf{z}_0 \times (\mathbf{p}_{l_2} - \mathbf{p}_0) & \mathbf{z}_1 \end{array} \right] = \left[\begin{array}{cc} -a_1 s_1 - d_2 c_1 & -s_1 \\ a_1 c_1 - d_2 s_1 & c_1 \\ 1 & 0 \end{array} \right]$$

For the above computations, we can make reference to the following picture:



and to the following auxiliary vectors:

$$\mathbf{p}_{l_1} = \left[\begin{array}{c} l_1 c_1 \\ l_1 s_1 \\ \star \end{array} \right], \mathbf{p}_{l_2} = \left[\begin{array}{c} a_1 c_1 - d_2 s_1 \\ a_1 s_1 + d_2 c_1 \\ d_1 \end{array} \right], \mathbf{p}_1 = \left[\begin{array}{c} a_1 c_1 \\ a_1 s_1 \\ d_1 \end{array} \right], \mathbf{z}_1 = \left[\begin{array}{c} -s_1 \\ c_1 \\ 0 \end{array} \right]$$

The inertia matrix can be computed now:

$$\begin{aligned} \mathbf{B}(\mathbf{q}) &= m_1 \mathbf{J}_P^{(l_1)^T} \mathbf{J}_P^{(l_1)} + I_1 \mathbf{J}_O^{(l_1)^T} \mathbf{J}_O^{(l_1)} + m_2 \mathbf{J}_P^{(l_2)^T} \mathbf{J}_P^{(l_2)} + \\ &= m_1 \left[\begin{array}{cc} l_1^2 & 0 \\ 0 & 0 \end{array} \right] + I_1 \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] + m_2 \left[\begin{array}{cc} a_1^2 + d_2^2 & a_1 \\ a_1 & 1 \end{array} \right] \\ &= \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{12} & b_{22} \end{array} \right] \end{aligned}$$

where:

$$\begin{aligned} b_{11} &= m_1 l_1^2 + I_1 + m_2 \left(a_1^2 + d_2^2 \right) \\ b_{12} &= m_2 a_1 \\ b_{22} &= m_2 \end{aligned}$$

2. Compute the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ of the Coriolis and centrifugal terms[1] for this manipulator.

   The only derivative in the Christoffel symbols which is different from zero is:

$$\frac{\partial b_{11}}{\partial q_2} = 2m_2 d_2$$

   therefore

$$\begin{array}{cc} c_{111} = 0 & c_{211} = -\frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = -m_2 d_2 \\ c_{112} = c_{121} = \frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = m_2 d_2 & c_{212} = c_{221} = 0 \\ c_{112} = 0 & c_{222} = 0 \end{array}$$

   The matrix of the Coriolis and centrifugal terms is thus:

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

   where:

$$\begin{aligned} c_{11} &= c_{111}\dot{q}_1 + c_{112}\dot{q}_2 = m_2 d_2 \dot{d}_2 \\ c_{12} &= c_{121}\dot{q}_1 + c_{122}\dot{q}_2 = m_2 d_2 \dot{\vartheta}_1 \\ c_{21} &= c_{211}\dot{q}_1 + c_{212}\dot{q}_2 = -m_2 d_2 \dot{\vartheta}_1 \\ c_{22} &= c_{221}\dot{q}_1 + c_{222}\dot{q}_2 = 0 \end{aligned}$$

3. Write the complete dynamic model for this manipulator and specify whether this model depends on both joint positions, both joint velocities, and both joint accelerations.

   Clearly the manipulator is not affected by gravitational effects. The model is then formed by the equation:

$$\mathbf{B}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\,\dot{\mathbf{q}} = \tau$$

   which corresponds to the scalar equations:

$$\begin{aligned} \left(m_1 l_1^2 + I_1 + m_2\left(a_1^2 + d_2^2\right)\right)\ddot{\vartheta}_1 + m_2 a_1 \ddot{d}_2 + 2m_2 d_2 \dot{\vartheta}_1 \dot{d}_2 &= \tau_1 \\ m_2 a_1 \ddot{\vartheta}_1 + m_2 \ddot{d}_2 - m_2 d_2 \dot{\vartheta}_1^2 &= \tau_2 \end{aligned}$$

   The model clearly depends on both the joint velocities and the acceleration, however only on the joint variable d_2 (and not on the joint variable ϑ_1).

[1] The general expression of the Christoffel symbols is $c_{ijk} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)$

4. Show that the model obtained in the previous step is linear with respect to a set of dynamic parameters. Is it possible through suitable experiments to identify the value of the mass of the first link m_1?

   The model can be written in the form:

$$\mathbf{Y}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right)\mathbf{\Pi} = \tau$$

   with:

$$\mathbf{\Pi} = \begin{bmatrix} m_1 l_1^2 + I_1 \\ m_2 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} \ddot{\vartheta}_1 & \left(a_1^2 + d_2^2\right)\ddot{\vartheta}_1 + a_1\ddot{d}_2 + 2d_2\dot{\vartheta}_1\dot{d}_2 \\ 0 & a_1\ddot{\vartheta}_1 + \ddot{d}_2 - d_2\dot{\vartheta}_1^2 \end{bmatrix}$$

   The mass of the first link contributes to the dynamic model only within the expression $m_1 l_1^2 + I_1$. It is therefore not possible to identify it with experiments.

EXERCISE 2

1. Write the parametric expression (in terms of a natural coordinate s) of a segment in space, used for planning a linear path.

   The general expression of the segment parameterized in the natural coordinate is:

$$\mathbf{p}(s) = \mathbf{p}_i + \frac{s}{\|\mathbf{p}_f - \mathbf{p}_i\|}\left(\mathbf{p}_f - \mathbf{p}_i\right)$$

2. Show that, in the general case, the absolute value of the time derivative of the natural coordinate s is the norm of the linear velocity of the end-effector.

   Taking the time derivative of the position (which is the linear velocity vector), we obtain

$$\dot{\mathbf{p}} = \frac{d\mathbf{p}}{ds}\dot{s} = \dot{s}\mathbf{t}$$

   where $\mathbf{t}$ is the tangent unit vector. Clearly $\|\dot{\mathbf{p}}\| = |\dot{s}|$

3. Consider now the planning along a linear path. Assume that the length of the segment to cover is 0.5 m and that the maximum linear velocity of the end effector is 1.5 m/s. Compute the minimum positioning time, if a cycloidal dependence on time [2] of the natural coordinate is used.

   If $h = 0.5$ is the total displacement and T is the travel time, it is:

$$\dot{s}_{\max} = \frac{h}{T}\sigma'_{\max}$$

[2]The normalized expression of a cycloidal trajectory is $\sigma(\tau) = \tau - \frac{1}{2\pi}\sin\left(2\pi\tau\right)$

Since:

$$\sigma'(\tau) = 1 - \cos(2\pi\tau)$$

we have:

$$\sigma'_{\max} = \sigma'(0.5) = 2$$

and then:

$$\dot{s}_{\max} = 2\frac{h}{T}$$

At this point we impose the inequality:

$$2\frac{0.5}{T} < 1.5$$

which implies:

$$T > \frac{1}{1.5} = 0.66$$

4. Explain what is an artificial potential method in the context of path planning with obstacle avoidance. What is a possible issue with this method?

   In the artificial potential method, the motion of the point that represents the robot in configuration space is influenced by a potential field U. This field is obtained as the superposition of an attractive potential to the goal and a repulsive potential from the obstacles. At each configuration $\mathbf{q}$ the artificial force generated by the potential is defined as the negative gradient $-\nabla U(\mathbf{q})$ of the potential.

   A possible issue is that the method can incur in local minima.

EXERCISE 3

Consider a robot that uses a camera.

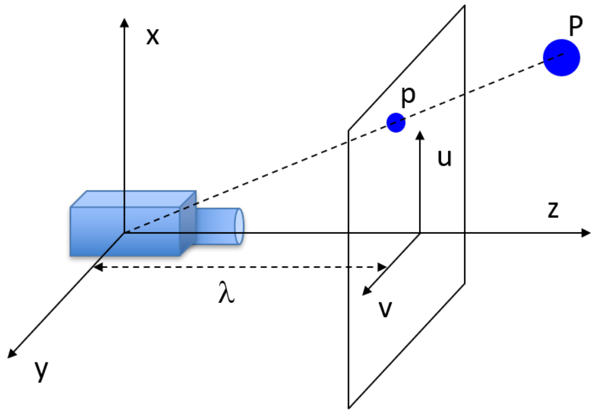
1. Explain what are the extrinsic and the intrinsic calibrations, making in particular reference to the notion of camera intrinsic matrix.

   The extrinsic calibration is the determination of the extrinsic parameters of the camera, like the position and the orientation of the camera with respect to a reference frame. The intrinsic calibration is the determination of the intrinsic parameters of the camera (like the focal length λ) as well as of some additional parameters. The intrinsic parameters are usually organized in a matrix (camera intrinsic matrix):

$$\mathbf{K} = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

   where c_x and c_y are the coordinates of the optical center, f_x and f_y are the ratios between the focal length and the size (along x and y) of the pixel, s is a skew parameter.

2. With reference to the following sketch, define what an image feature is and write the equations of the perspective projection method.



The image feature is the coding of any information that can be retrieved from an image, for example the two coordinates of a point in the image plane. The equations of the perspective projection can be written as:

$$\xi = \begin{bmatrix} u \\ v \end{bmatrix} = \frac{\lambda}{Z} \begin{bmatrix} X \\ Y \end{bmatrix}$$

3. Define the interaction matrix and the image Jacobian for a vision-based robotic system, in terms of the quantities that each of the two matrices relate.

   The interaction matrix relates the linear and angular velocities of the camera to the velocity in the image plane:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \mathbf{L} \begin{bmatrix} \dot{\mathbf{O}}_c \\ \omega_c \end{bmatrix}$$

   The image Jacobian relates the joint velocities of the robot to the velocity in the image plane:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \mathbf{J}_I \dot{\mathbf{q}}$$

4. In the process of deriving the interaction matrix, the following equation is used:

$$\dot{\mathbf{P}} = -\omega_c \times \mathbf{P} - \dot{\mathbf{O}}_c$$

   Explain the meaning of all symbols used in this equation. The equations of the perspective projection are used to elaborate this equation: explain what are the variables of the previous equation that enter the perspective projection equations.

   In the previous equation, all vectors are referred to the camera frame. $\mathbf{P}$ and $\dot{\mathbf{P}}$ are the position and linear velocity, respectively, of a point in the camera frame, $\dot{\mathbf{O}}_c$ and ω_c are the linear and angular velocities of the camera, respectively. The perspective projection relates the element of vector **P** with the image features and the focal length.