# Control of Industrial Robots 

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SOLUTION

## Control of Industrial Robots Prof. Paolo Rocco

## EXERCISE 1

1. Consider the manipulator sketched in the picture, where the mass of the second link is assumed to be concentrated at the end-effector:


Find the expression of the inertia matrix $\mathbf{B}(\mathbf{q})$ of the manipulator.

Denavit-Hartenberg frames can be defined as sketched in this picture:


Computations of the Jacobians:

Link 1

$$
\begin{gathered}
\mathbf{J}_{P}^{\left(l_{1}\right)}=\left[\begin{array}{ll}
\mathbf{j}_{P_{1}}^{\left(l_{1}\right)} & \mathbf{0}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{z}_{0} \times\left(\mathbf{p}_{l_{1}}-\mathbf{p}_{0}\right) & \mathbf{0}
\end{array}\right]=\left[\begin{array}{cc}
-l_{1} s_{1} & 0 \\
l_{1} c_{1} & 0 \\
0 & 0
\end{array}\right] \\
\mathbf{J}_{O}^{\left(l_{1}\right)}=\left[\begin{array}{ll}
\mathbf{j}_{O_{1}}^{\left(l_{1}\right)} & \mathbf{0}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{z}_{0} & \mathbf{0}
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0 \\
1 & 0
\end{array}\right]
\end{gathered}
$$

Link 2

$$
\mathbf{J}_{P}^{\left(l_{2}\right)}=\left[\begin{array}{ll}
\mathbf{j}_{P_{1}}^{\left(l_{2}\right)} & \mathbf{j}_{P_{2}}^{\left(l_{2}\right)}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{z}_{0} \times\left(\mathbf{p}_{l_{2}}-\mathbf{p}_{0}\right) & \mathbf{z}_{1}
\end{array}\right]=\left[\begin{array}{cc}
-a_{1} s_{1}-d_{2} c_{1} & -s_{1} \\
a_{1} c_{1}-d_{2} s_{1} & c_{1} \\
1 & 0
\end{array}\right]
$$

For the above computations, we can make reference to the following picture:

and to the following auxiliary vectors:

$$
\mathbf{p}_{l_{1}}=\left[\begin{array}{c}
l_{1} c_{1} \\
l_{1} s_{1} \\
\star
\end{array}\right], \mathbf{p}_{l_{2}}=\left[\begin{array}{c}
a_{1} c_{1}-d_{2} s_{1} \\
a_{1} s_{1}+d_{2} c_{1} \\
d_{1}
\end{array}\right], \mathbf{p}_{1}=\left[\begin{array}{c}
a_{1} c_{1} \\
a_{1} s_{1} \\
d_{1}
\end{array}\right], \mathbf{z}_{1}=\left[\begin{array}{c}
-s_{1} \\
c_{1} \\
0
\end{array}\right]
$$

The inertia matrix can be computed now:

$$
\begin{aligned}
\mathbf{B}(\mathbf{q}) & =m_{1} \mathbf{J}_{P}^{\left(l_{1}\right)^{T}} \mathbf{J}_{P}^{\left(l_{1}\right)}+I_{1} \mathbf{J}_{O}^{\left(l_{1}\right)^{T}} \mathbf{J}_{O}^{\left(l_{1}\right)}+m_{2} \mathbf{J}_{P}^{\left(l_{2}\right)^{T}} \mathbf{J}_{P}^{\left(l_{2}\right)}+ \\
& =m_{1}\left[\begin{array}{cc}
l_{1}^{2} & 0 \\
0 & 0
\end{array}\right]+I_{1}\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]+m_{2}\left[\begin{array}{cc}
a_{1}^{2}+d_{2}^{2} & a_{1} \\
a_{1} & 1
\end{array}\right] \\
& =\left[\begin{array}{ll}
b_{11} & b_{12} \\
b_{12} & b_{22}
\end{array}\right]
\end{aligned}
$$

where:

$$
\begin{aligned}
b_{11} & =m_{1} l_{1}^{2}+I_{1}+m_{2}\left(a_{1}^{2}+d_{2}^{2}\right) \\
b_{12} & =m_{2} a_{1} \\
b_{22} & =m_{2}
\end{aligned}
$$

2. Compute the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ of the Coriolis and centrifugal terms ${ }^{1}$ for this manipulator.

The only derivative in the Christoffel symbols which is different from zero is:

$$
\frac{\partial b_{11}}{\partial q_{2}}=2 m_{2} d_{2}
$$

therefore

$$
\begin{array}{cc}
c_{111}=0 & c_{211}=-\frac{1}{2} \frac{\partial b_{11}}{\partial q_{2}}=-m_{2} d_{2} \\
c_{112}=c_{121}=\frac{1}{2} \frac{\partial b_{11}}{\partial q_{2}}=m_{2} d_{2} & c_{212}=c_{221}=0 \\
c_{112}=0 & c_{222}=0
\end{array}
$$

The matrix of the Coriolis and centrifugal terms is thus:

$$
\mathbf{C}=\left[\begin{array}{ll}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{array}\right]
$$

where:

$$
\begin{aligned}
& c_{11}=c_{111} \dot{q}_{1}+c_{112} \dot{q}_{2}=m_{2} d_{2} \dot{d}_{2} \\
& c_{12}=c_{121} \dot{q}_{1}+c_{122} \dot{q}_{2}=m_{2} d_{2} \dot{\vartheta}_{1} \\
& c_{21}=c_{211} \dot{q}_{1}+c_{212} \dot{q}_{2}=-m_{2} d_{2} \dot{\vartheta}_{1} \\
& c_{22}=c_{221} \dot{q}_{1}+c_{222} \dot{q}_{2}=0
\end{aligned}
$$

3. Write the complete dynamic model for this manipulator.

Clearly the manipulator is not affected by gravitational effects. The model is then formed by the equation:

$$
\mathbf{B}(\mathbf{q}) \ddot{\mathbf{q}}+\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}=\tau
$$

which corresponds to the scalar equations:

$$
\begin{aligned}
\left(m_{1} l_{1}^{2}+I_{1}+m_{2}\left(a_{1}^{2}+d_{2}^{2}\right)\right) \ddot{\vartheta}_{1}+m_{2} a_{1} \ddot{d}_{2}+2 m_{2} d_{2} \dot{\vartheta}_{1} \dot{d}_{2} & =\tau_{1} \\
m_{2} a_{1} \ddot{\vartheta}_{1}+m_{2} \ddot{d}_{2}-m_{2} d_{2} \dot{\theta}_{1}^{2} & =\tau_{2}
\end{aligned}
$$

4. Show that the model obtained in the previous step is linear with respect to a set of dynamic parameters.

The model can be written in the form:

[^0]$$
\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) \boldsymbol{\Pi}=\tau
$$
with:
\[

$$
\begin{gathered}
\boldsymbol{\Pi}=\left[\begin{array}{c}
m_{1} l_{1}^{2}+I_{1} \\
m_{2}
\end{array}\right] \\
\mathbf{Y}=\left[\begin{array}{cc}
\ddot{\vartheta}_{1} & \left(a_{1}^{2}+d_{2}^{2}\right) \ddot{\vartheta}_{1}+a_{1} \ddot{d}_{2}+2 d_{2} \dot{\vartheta}_{1} \dot{d}_{2} \\
0 & a_{1} \ddot{\vartheta}_{1}+\ddot{d}_{2}-d_{2} \dot{\vartheta}_{1}^{2}
\end{array}\right]
\end{gathered}
$$
\]

## EXERCISE 2

1. The parametric form of a harmonic trajectory for kinematic scaling is given by:

$$
\sigma(\tau)=\frac{1}{2}(1-\cos (\pi \tau))
$$

Find the expressions of the maximum velocity and maximum acceleration for such trajectory in terms of the positioning time $T$ and the total displacement $h$.

The first and second derivatives of the parametric form are given by:

$$
\begin{aligned}
\sigma^{\prime}(\tau) & =\frac{\pi}{2} \sin (2 \pi \tau) \\
\sigma^{\prime \prime}(\tau) & =\frac{\pi^{2}}{2} \cos (2 \pi \tau)
\end{aligned}
$$

The expressions of the maximum velocity and acceleration can be obtained as:

$$
\begin{aligned}
\dot{q}_{\max } & =\frac{h}{T} \sigma_{\max }^{\prime}(\tau)=\frac{h}{T} \sigma^{\prime}(0.5)=\frac{\pi h}{2 T} \\
\ddot{q}_{\max } & =\frac{h}{T^{2}} \sigma_{\max }^{\prime \prime}(\tau)=\frac{h}{T^{2}} \sigma^{\prime \prime}(0)=\frac{\pi^{2} h}{2 T^{2}}
\end{aligned}
$$

2. Consider the design of a harmonic trajectory from $q_{i}=10$ to $q_{f}=20$, with $\dot{q}_{\max }=20$ and $\ddot{q}_{\max }=20$. Find the minimum positioning time.

The total displacement is $h=10$. Therefore:

$$
\begin{aligned}
\frac{\pi h}{2 T}<20 & \Rightarrow T>\frac{\pi}{4} \\
\frac{\pi^{2} h}{2 T^{2}}<20 & \Rightarrow T>\frac{\pi}{2}
\end{aligned}
$$

The minimum positioning time is then $T_{\min }=\frac{\pi}{2}=1.57$
3. For the harmonic trajectory computed in this exercise, sketch the plot of the speed $\dot{q}(t)$, assuming that the trajectory starts at time $t_{i}=0$.

The speed has the typical bell-shaped aspect and takes the maximum value in the middle time instant:


Notice that the maximum possible value of the speed $\dot{q}_{\max }=20$ is not reached since the positioning time is twice as much as the time needed to reach that value. The maximum value reached by the speed is then $0.5 \dot{q}_{\max }=10$.
4. Still for the harmonic trajectory computed in this exercise, is the acceleration continuous in all time instants, including the initial and final times? What kind of issues discontinuities in the acceleration profile might imply in robotics?

The harmonic trajectory has a discontinuous acceleration at the initial and finale instants. Discontinuities in the acceleration might excite resonant dynamics in the robot and then induce vibrations.

## EXERCISE 3

1. Consider an interaction task of a manipulator, with a frictionless and rigid surface, as in this picture:


Express the natural and the artificial constraints for this problem, and specify the selection matrix.

The natural constraints and artificial constraints can be easily identified:

| Natural constraints | Artificial constraints |
| :---: | :---: |
| $f_{x}^{c}$ | $\dot{p}_{x}^{c}$ |
| $f_{y}^{c}$ | $\dot{p}_{y}^{c}$ |
| $\dot{p}_{z}^{c}$ | $f_{z}^{c}$ |
| $\omega_{x}^{c}$ | $\mu_{x}^{c}$ |
| $\omega_{y}^{c}$ | $\mu_{y}^{c}$ |
| $\mu_{z}^{c}$ | $\omega_{z}^{c}$ |

The selection matrix is thus:

$$
\Sigma=\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

2. Sketch the block diagram of a hybrid force-position controller. What are possible sources of inconsistency in the adoption of such scheme?

The block diagram of a hybrid force-position controller is as follows:


Possible sources of inconsistency are friction at the contact (a force is detected in a nominally free direction), compliance in the robot structure and/or at the contact (a displacement is detected in a direction which is nominally constrained in motion), uncertainty in the environment geometry at the contact.
3. Explain what an implicit force controller is and why it might be convenient with respect to an explicit solution.

An implicit force control is closed around the position control loops. This is usually the only viable solution to implement force control, since the reliable and industrially safe position controllers cannot be bypassed.
4. Suppose now that along the translational $z$ direction an implicit force controller has to be designed. Sketch the block diagram of such controller and design it taking a bandwidth of $20 \mathrm{rad} / \mathrm{s}$.

The block diagram of an implicit force controller in case of rigid surface is sketched in the picture:

where $R(s)$ is the transfer function of the position controller. If we assume a PID position controller:

$$
R(s)=\frac{K_{D} s^{2}+K_{P} s+K_{I}}{s}
$$

The partial compensator of such controller is:

$$
C(s)=\frac{1}{K_{D} s^{2}+K_{P} s+K_{I}}
$$

If we select a PI controller on the force error:

$$
R_{f}(s)=k_{p f}+\frac{k_{i f}}{s}
$$

the loop transfer function becomes:

$$
L_{f}(s)=\frac{s k_{p f}+k_{i f}}{s^{2}}
$$

Since the high frequency approximation of such transfer function is $k_{p f} / s$ we can set $k_{p f}=20$ (equal to the required bandwidth. The zero of the controller can be set at a lower frequency range, for example $k_{i f} / k_{p f}=2$, which yields $k_{i f}=40$.


[^0]:    ${ }^{1}$ The general expression of the Christoffel symbols is $c_{i j k}=\frac{1}{2}\left(\frac{\partial b_{i j}}{\partial q_{k}}+\frac{\partial b_{i k}}{\partial q_{j}}-\frac{\partial b_{j k}}{\partial q_{i}}\right)$

