# Control of Industrial Robots 

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SOLUTION

## Control of Industrial Robots <br> Prof. Paolo Rocco

## EXERCISE 1

Consider the manipulator sketched in the picture:


1. Find the expression of the inertia matrix $\mathbf{B}(\mathbf{q})$ of the manipulator ${ }^{1}$

Denavit-Hartenberg frames can be defined as sketched in this picture:


Computations of the Jacobians:
${ }^{1}$ The cross product between vector $a=\left[\begin{array}{l}a_{1} \\ a_{2} \\ a_{3}\end{array}\right]$ and $b=\left[\begin{array}{c}b_{1} \\ b_{2} \\ b_{3}\end{array}\right]$ is $c=a \times b=\left[\begin{array}{l}a_{2} b_{3}-a_{3} b_{2} \\ a_{3} b_{1}-a_{1} b_{3} \\ a_{1} b_{2}-a_{2} b_{1}\end{array}\right]$

Link 1

$$
\mathbf{J}_{P}^{\left(l_{1}\right)}=\left[\begin{array}{cc}
\mathbf{j}_{P_{1}}^{\left(l_{1}\right)} & \mathbf{0}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{z}_{0} & \mathbf{0}
\end{array}\right]=\left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
1 & 0
\end{array}\right]
$$

Link 2

$$
\begin{gathered}
\mathbf{J}_{P}^{\left(l_{2}\right)}=\left[\begin{array}{ll}
\mathbf{j}_{P_{1}}^{\left(l_{2}\right)} & \mathbf{j}_{P_{2}}^{\left(l_{2}\right)}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{z}_{0} & \mathbf{z}_{1} \times\left(\mathbf{p}_{l_{2}}-\mathbf{p}_{1}\right)
\end{array}\right]=\left[\begin{array}{cc}
0 & -l_{2} s_{2} \\
0 & 0 \\
1 & l_{2} c_{2}
\end{array}\right] \\
\mathbf{J}_{O}^{\left(l_{2}\right)}=\left[\begin{array}{ll}
\mathbf{j}_{O_{1}}^{\left(l_{2}\right)} & \mathbf{j}_{O_{2}}^{\left(l_{2}\right)}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{0} & \mathbf{z}_{1}
\end{array}\right]=\left[\begin{array}{cc}
0 & 0 \\
0 & -1 \\
0 & 0
\end{array}\right]
\end{gathered}
$$

For the above computations, we can make reference to the following picture:

and to the following auxiliary vectors:

$$
\mathbf{p}_{l_{2}}=\left[\begin{array}{c}
-a_{1}+l_{2} c_{2} \\
0 \\
d_{1}+l_{2} s_{2}
\end{array}\right], \mathbf{p}_{1}=\left[\begin{array}{c}
-a_{1} \\
0 \\
d_{1}
\end{array}\right], \mathbf{z}_{1}=\left[\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right]
$$

The inertia matrix can be computed now:

$$
\begin{aligned}
\mathbf{B}(\mathbf{q}) & =m_{1} \mathbf{J}_{P}^{\left(l_{1}\right)^{T}} \mathbf{J}_{P}^{\left(l_{1}\right)}+m_{2} \mathbf{J}_{P}^{\left(l_{2}\right)^{T}} \mathbf{J}_{P}^{\left(l_{2}\right)}+I_{2} \mathbf{J}_{O}^{\left(l_{2}\right)^{T}} \mathbf{J}_{O}^{\left(l_{2}\right)} \\
& =m_{1}\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]+m_{2}\left[\begin{array}{cc}
1 & l_{2} c_{2} \\
l_{2} c_{2} & l_{2}^{2}
\end{array}\right]+I_{2}\left[\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right] \\
& =\left[\begin{array}{ll}
b_{11} & b_{12} \\
b_{12} & b_{22}
\end{array}\right]
\end{aligned}
$$

where:

$$
\begin{aligned}
b_{11} & =m_{1}+m_{2} \\
b_{12} & =m_{2} l_{2} c_{2} \\
b_{22} & =m_{2} l_{2}^{2}+I_{2}
\end{aligned}
$$

2. Compute the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ of the Coriolis and centrifugal terms ${ }^{2}$ for this manipulator.

The only derivative in the Christoffel symbols which is different from zero is:

$$
\frac{\partial b_{12}}{\partial q_{2}}=\frac{\partial b_{21}}{\partial q_{2}}=-m_{2} l_{2} s_{2}
$$

therefore

$$
\begin{array}{cc}
c_{111}=0 & c_{211}=0 \\
c_{112}=c_{121}=0 & c_{212}=c_{221}=\frac{1}{2}\left(\frac{\partial b_{21}}{\partial q_{2}}-\frac{\partial b_{12}}{\partial q_{2}}\right)=0 \\
c_{122}=\frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_{2}}+\frac{\partial b_{21}}{\partial q_{2}}\right)=-m_{2} l_{2} s_{2} & c_{222}=0
\end{array}
$$

The matrix of the Coriolis and centrifugal terms is thus:

$$
\mathbf{C}=\left[\begin{array}{ll}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{array}\right]
$$

where:

$$
\begin{aligned}
& c_{11}=c_{111} \dot{q}_{1}+c_{112} \dot{q}_{2}=0 \\
& c_{12}=c_{121} \dot{q}_{1}+c_{122} \dot{q}_{2}=-m_{2} l_{2} s_{2} \dot{\vartheta}_{2} \\
& c_{21}=c_{211} \dot{q}_{1}+c_{212} \dot{q}_{2}=0 \\
& c_{22}=c_{221} \dot{q}_{1}+c_{222} \dot{q}_{2}=0
\end{aligned}
$$

3. Check that matrix $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})=\dot{\mathbf{B}}(\mathbf{q})-2 \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew symmetric.

We have that:

$$
\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})=\dot{\mathbf{B}}(\mathbf{q})-2 \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})=\left[\begin{array}{cc}
0 & -m_{2} l_{2} s_{2} \dot{\vartheta}_{2} \\
-m_{2} l_{2} s_{2} \dot{\vartheta}_{2} & 0
\end{array}\right]-2\left[\begin{array}{cc}
0 & -m_{2} l_{2} s_{2} \dot{\vartheta}_{2} \\
0 & 0
\end{array}\right]=\left[\begin{array}{cc}
0 & m_{2} l_{2} s_{2} \dot{\vartheta}_{2} \\
-m_{2} l_{2} s_{2} \dot{\vartheta}_{2} & 0
\end{array}\right]
$$

which is a skew-symmetric matrix.
4. For a generic manipulator, ignoring the gravitational terms and exploiting the skew symmetry of matrix $\mathbf{N}$, obtain an expression of the derivative with respect to time of the kinetic energy.

The kinetic energy is:

$$
T=\frac{1}{2} \dot{\mathbf{q}}^{T} \mathbf{B}(\mathbf{q}) \dot{\mathbf{q}}
$$

[^0]Its time derivative is:

$$
\frac{d T}{d t}=\dot{\mathbf{q}}^{T} \mathbf{B}(\mathbf{q}) \ddot{\mathbf{q}}+\frac{1}{2} \dot{\mathbf{q}}^{T} \dot{\mathbf{B}}(\mathbf{q}) \dot{\mathbf{q}}
$$

Exploiting the dynamic model we have:

$$
\frac{d T}{d t}=\dot{\mathbf{q}}^{T}[-\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})+\tau]+\frac{1}{2} \dot{\mathbf{q}}^{T} \dot{\mathbf{B}}(\mathbf{q}) \dot{\mathbf{q}}=\frac{1}{2} \dot{\mathbf{q}}^{T}[\dot{\mathbf{B}}-2 \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})] \dot{\mathbf{q}}+\dot{\mathbf{q}}^{T} \tau
$$

Thanks to the skew symmetry of the matrix, we finally obtain:

$$
\frac{d T}{d t}=\dot{\mathbf{q}}^{T} \tau
$$

## EXERCISE 2

Consider a kinematically redundant manipulator.

1. Write the general expression of the solutions of the inverse kinematics problem at velocity level.

Given a set of desired task variables $\mathbf{r}_{d}$, such that:

$$
\dot{\mathbf{r}}_{d}=\mathbf{J} \dot{\mathbf{q}}
$$

where $\mathbf{J}$ is the Jacobian matrix, the general expression of the solution of the inverse kinematics is:

$$
\dot{\mathbf{q}}=\mathbf{J}^{\sharp} \dot{\mathbf{r}}_{d}+\mathbf{P} \dot{\mathbf{q}}_{0}
$$

$\mathbf{J}^{\sharp}$ is the pseudo-inverse of the Jacobian, defined as:

$$
\mathbf{J}^{\sharp}=\mathbf{J}^{T}\left(\mathbf{J J}^{T}\right)^{-1}
$$

The term $\mathbf{P} \dot{\mathbf{q}}_{0}$ defines the null-space motions: $\mathbf{P}$ is a matrix that projects a generic joint velocity $\dot{\mathbf{q}}_{0}$ in the null space of the Jacobian and takes the expression:

$$
\mathbf{P}=\mathbf{I}_{n}-\mathbf{J}^{\sharp} \mathbf{J}
$$

2. Express the solution in the form that includes a closed loop correction (kinematic control) and explain why this correction is used.

The solution with closed-loop correction can be written as:

$$
\dot{\mathbf{q}}=\mathbf{J}^{\sharp}\left[\dot{\mathbf{r}}_{d}+\mathbf{K}\left(\mathbf{r}_{d}-\mathbf{r}\right)\right]+\mathbf{P} \dot{\mathbf{q}}_{0}
$$

where:

$$
\mathbf{r}=\mathbf{f}(\mathbf{q})
$$

is obtained through direct kinematics. The correction is used to recover errors with respect to an assigned task $\mathbf{r}_{d}$ due to initial mismatches, drifts, inaccuracies of the solution
3. Consider now the motion of the end effector along a linear path. Assigning to the natural coordinate $s$ a cubic dependence on time, derive the expressions of the maximum speed and the maximum acceleration as functions of the displacement $h$ and the positioning time $T$.

We express the natural coordinate in a terms of a normalized displacement $\sigma$ and a normalized time $\tau$ :

$$
s=s_{i}+h \sigma(\tau)
$$

The expression of the normalized displacement is:

$$
\sigma=a_{0}+a_{1} \tau+a_{2} \tau^{2}+a_{3} \tau^{3}
$$

With the boundary conditions $\sigma(0)=0, \sigma(1)=1, \sigma^{\prime}(0)=0, \sigma^{\prime}(1)=0$, the expression of $\sigma$ and of its first derivatives is:

$$
\begin{aligned}
\sigma(\tau) & =3 \tau^{2}-2 \tau^{3} \\
\sigma^{\prime}(\tau) & =6 \tau-6 \tau^{2} \\
\sigma^{\prime \prime}(\tau) & =6-12 \tau
\end{aligned}
$$

The maximum values of such derivatives are then:

$$
\begin{aligned}
\sigma_{\max }^{\prime} & =\sigma^{\prime}(0.5)=\frac{3}{2} \\
\sigma_{\max }^{\prime \prime} & =\sigma^{\prime \prime}(0)=6
\end{aligned}
$$

which yields to the expressions of the maximum speed and acceleration:

$$
\begin{aligned}
\dot{s}_{\max } & =\frac{3}{2} \frac{h}{T} \\
\ddot{s}_{\max } & =6 \frac{h}{T^{2}}
\end{aligned}
$$

4. Assume that the length of the segment to cover is 1 m , the maximum linear velocity of the end effector is $2 \mathrm{~m} / \mathrm{s}$ and the maximum linear acceleration $4 \mathrm{~m} / \mathrm{s}^{2}$. Compute the minimum positioning time, adopting a cubic dependence on time.

The expressions of the maximum speed and acceleration obtained previously imply the following inequalities on the positioning time:

$$
\begin{aligned}
T & \geq \frac{3 h}{2 \dot{s}_{\max }}=\frac{3}{4}=0.75 \\
T & \geq \sqrt{\frac{6 h}{\frac{\dot{s}_{\max }}{}}}=\sqrt{\frac{3}{2}}=1.22
\end{aligned}
$$

The minimum positioning time is then 1.22

## EXERCISE 3

Consider a robot that uses a camera.

1. Explain what are the extrinsic and the intrinsic calibrations, making in particular reference to the notion of camera intrinsic matrix.

The extrinsic calibration is the determination of the extrinsic parameters of the camera, like the position and the orientation of the camera with respect to a reference frame. The intrinsic calibration is the determination of the intrinsic parameters of the camera (like the focal length $\lambda$ ) as well as of some additional parameters. The intrinsic parameters are usually organized in a matrix (camera intrinsic matrix):

$$
\mathbf{K}=\left[\begin{array}{ccc}
f_{x} & s & c_{x} \\
0 & f_{y} & c_{y} \\
0 & 0 & 1
\end{array}\right]
$$

where $c_{x}$ and $c_{y}$ are the coordinates of the optical center, $f_{x}$ and $f_{y}$ are the ratios between the focal length and the size (along x and y ) of the pixel, $s$ is a skew parameter.
2. With reference to the following sketch, define what an image feature is and write the equations of the perspective projection method.


The image feature is the coding of any information that can be retrieved from an image, for example the two coordinates of a point in the image plane. The equations of the perspective projection can be written as:

$$
\xi=\left[\begin{array}{l}
u \\
v
\end{array}\right]=\frac{\lambda}{Z}\left[\begin{array}{l}
X \\
Y
\end{array}\right]
$$

3. Define the interaction matrix and the image Jacobian for a vision-based robotic system, in terms of the quantities that each of the two matrices relate.

The interaction matrix relates the linear and angular velocities of the camera to the velocity in the image plane:

$$
\left[\begin{array}{c}
\dot{u} \\
\dot{v}
\end{array}\right]=\mathbf{L}\left[\begin{array}{l}
\dot{\mathbf{O}}_{c} \\
\omega_{c}
\end{array}\right]
$$

The image Jacobian relates the joint velocities of the robot to the velocity in the image plane:

$$
\left[\begin{array}{c}
\dot{u} \\
\dot{v}
\end{array}\right]=\mathbf{J}_{I} \dot{\mathbf{q}}
$$

4. Consider now the following block diagram:


Is this a look-and-move or a visual servoing scheme? A position-based or an image-based scheme? Write an expression of the control law that can be used in this control scheme.

The scheme corresponds to a look-and-move image-based control scheme. The control law can be written as:

$$
\dot{\mathbf{q}}=\mathbf{J}_{I}^{\sharp}\left(\dot{\xi}_{d}+K\left(\xi_{d}-\xi\right)\right)+\left(\mathbf{I}-\mathbf{J}_{I}^{\sharp} \mathbf{J}_{I}\right) \dot{\mathbf{q}}_{0}
$$


[^0]:    ${ }^{2}$ The general expression of the Christoffel symbols is $c_{i j k}=\frac{1}{2}\left(\frac{\partial b_{i j}}{\partial q_{k}}+\frac{\partial b_{i k}}{\partial q_{j}}-\frac{\partial b_{j k}}{\partial q_{i}}\right)$

