Control of Industrial and Mobile Robots

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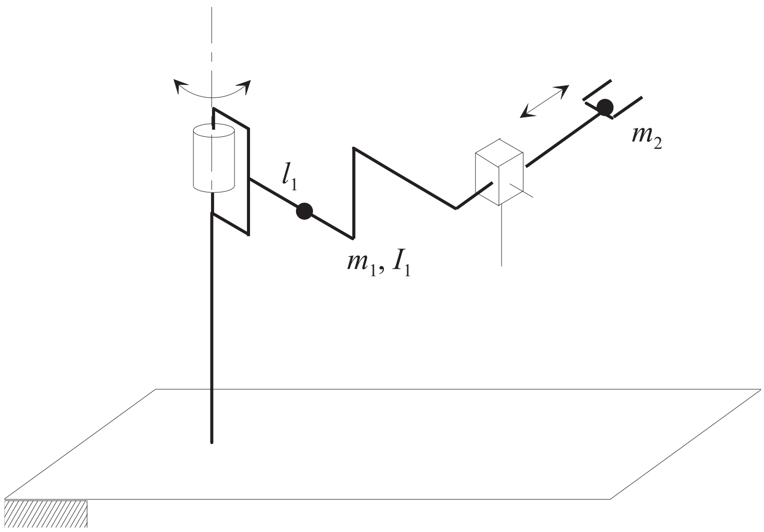
SOLUTION

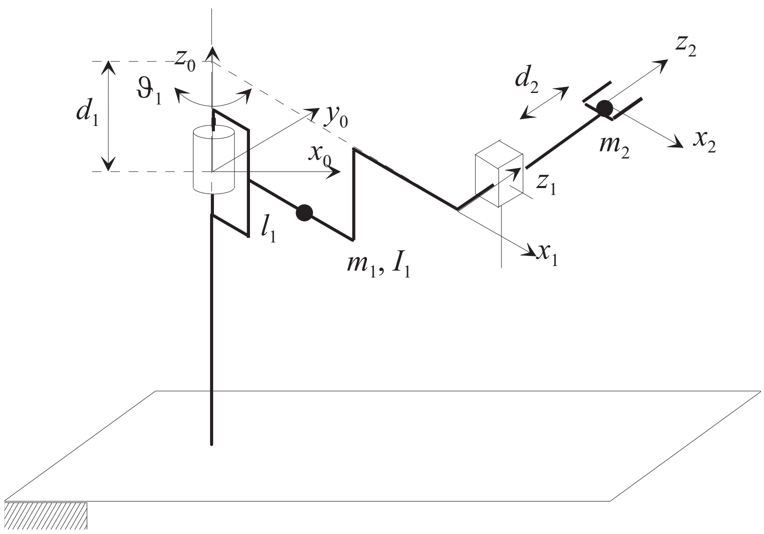
EXERCISE 1

1. Consider the manipulator sketched in the picture, where the mass of the second link is assumed to be concentrated at the end-effector:



Find the expression of the inertia matrix $\mathbf{B}(\mathbf{q})$ of the manipulator.

Denavit-Hartenberg frames can be defined as sketched in this picture:



Computations of the Jacobians:

Link 1

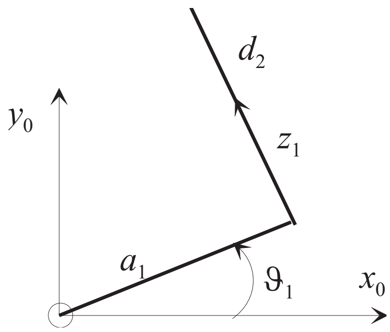
$$\mathbf{J}_P^{(l_1)} = \begin{bmatrix} \mathbf{j}_{P_1}^{(l_1)} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_{l_1} - \mathbf{p}_0) & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{J}_O^{(l_1)} = \begin{bmatrix} \mathbf{j}_{O_1}^{(l_1)} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Link 2

$$\mathbf{J}_P^{(l_2)} = \begin{bmatrix} \mathbf{j}_{P_1}^{(l_2)} & \mathbf{j}_{P_2}^{(l_2)} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_{l_2} - \mathbf{p}_0) & \mathbf{z}_1 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - d_2 c_1 & -s_1 \\ a_1 c_1 - d_2 s_1 & c_1 \\ 1 & 0 \end{bmatrix}$$

For the above computations, we can make reference to the following picture:



and to the following auxiliary vectors:

$$\mathbf{p}_{l_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ \star \end{bmatrix}, \mathbf{p}_{l_2} = \begin{bmatrix} a_1 c_1 - d_2 s_1 \\ a_1 s_1 + d_2 c_1 \\ d_1 \end{bmatrix}, \mathbf{p}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ d_1 \end{bmatrix}, \mathbf{z}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

The inertia matrix can be computed now:

$$\begin{aligned} \mathbf{B}(\mathbf{q}) &= m_1 \mathbf{J}_P^{(l_1)^T} \mathbf{J}_P^{(l_1)} + I_1 \mathbf{J}_O^{(l_1)^T} \mathbf{J}_O^{(l_1)} + m_2 \mathbf{J}_P^{(l_2)^T} \mathbf{J}_P^{(l_2)} + \\ &= m_1 \begin{bmatrix} l_1^2 & 0 \\ 0 & 0 \end{bmatrix} + I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} a_1^2 + d_2^2 & a_1 \\ a_1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \end{aligned}$$

where:

$$\begin{aligned} b_{11} &= m_1 l_1^2 + I_1 + m_2 \left(a_1^2 + d_2^2\right) \\ b_{12} &= m_2 a_1 \\ b_{22} &= m_2 \end{aligned}$$

2. Compute the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ of the Coriolis and centrifugal terms[1] for this manipulator.

The only derivative in the Christoffel symbols which is different from zero is:

$$\frac{\partial b_{11}}{\partial q_2} = 2m_2 d_2$$

therefore

$$\begin{array}{cc} c_{111} = 0 & c_{211} = -\frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = -m_2 d_2 \\ c_{112} = c_{121} = \frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = m_2 d_2 & c_{212} = c_{221} = 0 \\ c_{112} = 0 & c_{222} = 0 \end{array}$$

The matrix of the Coriolis and centrifugal terms is thus:

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where:

$$\begin{aligned} c_{11} &= c_{111}\dot{q}_1 + c_{112}\dot{q}_2 = m_2 d_2 \dot{d}_2 \\ c_{12} &= c_{121}\dot{q}_1 + c_{122}\dot{q}_2 = m_2 d_2 \dot{\vartheta}_1 \\ c_{21} &= c_{211}\dot{q}_1 + c_{212}\dot{q}_2 = -m_2 d_2 \dot{\vartheta}_1 \\ c_{22} &= c_{221}\dot{q}_1 + c_{222}\dot{q}_2 = 0 \end{aligned}$$

3. Write the complete dynamic model for this manipulator and specify whether this model depends on both joint positions, both joint velocities, and both joint accelerations.

Clearly the manipulator is not affected by gravitational effects. The model is then formed by the equation:

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \tau$$

which corresponds to the scalar equations:

$$\begin{aligned} \left(m_1 l_1^2 + I_1 + m_2\left(a_1^2 + d_2^2\right)\right)\ddot{\vartheta}_1 + m_2 a_1 \ddot{d}_2 + 2m_2 d_2 \dot{\vartheta}_1 \dot{d}_2 &= \tau_1 \\ m_2 a_1 \ddot{\vartheta}_1 + m_2 \ddot{d}_2 - m_2 d_2 \dot{\vartheta}_1^2 &= \tau_2 \end{aligned}$$

The model clearly depends on both the joint velocities and the acceleration, however only on the joint variable d_2 (and not on the joint variable ϑ_1).

[1] The general expression of the Christoffel symbols is $c_{ijk} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)$

4. Show that the model obtained in the previous step is linear with respect to a set of dynamic parameters. Is it possible through suitable experiments to identify the value of the mass of the first link m_1?

   The model can be written in the form:

$$\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\,\mathbf{\Pi} = \tau$$

   with:

$$\mathbf{\Pi} = \begin{bmatrix} m_1 l_1^2 + I_1 \\ m_2 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} \ddot{\vartheta}_1 & \left(a_1^2 + d_2^2\right)\ddot{\vartheta}_1 + a_1 \ddot{d}_2 + 2 d_2 \dot{\vartheta}_1 \dot{d}_2 \\ 0 & a_1 \ddot{\vartheta}_1 + \ddot{d}_2 - d_2 \dot{\vartheta}_1^2 \end{bmatrix}$$

   The mass of the first link contributes to the dynamic model only within the expression $m_1 l_1^2 + I_1$. It is therefore not possible to identify it with experiments.

EXERCISE 2

1. Write the parametric expression (in terms of a natural coordinate s) of a segment in space, used for planning a linear path.

   The general expression of the segment parameterized in the natural coordinate is:

$$\mathbf{p}(s) = \mathbf{p}_i + \frac{s}{\|\mathbf{p}_f - \mathbf{p}_i\|}\left(\mathbf{p}_f - \mathbf{p}_i\right)$$

2. Show that, in the general case, the absolute value of the time derivative of the natural coordinate s is the norm of the linear velocity of the end-effector.

   Taking the time derivative of the position (which is the linear velocity vector), we obtain

$$\dot{\mathbf{p}} = \frac{d\mathbf{p}}{ds}\dot{s} = \dot{s}\mathbf{t}$$

   where $\mathbf{t}$ is the tangent unit vector. Clearly $\|\dot{\mathbf{p}}\| = |\dot{s}|$

3. Consider now the planning along a linear path. Assume that the length of the segment to cover is 0.5 m and that the maximum linear velocity of the end effector is 1.5 m/s. Compute the minimum positioning time, if a cycloidal dependence on time [2] of the natural coordinate is used.

   If $h = 0.5$ is the total displacement and T is the travel time, it is:

$$\dot{s}_{\max} = \frac{h}{T}\sigma'_{\max}$$

[2]The normalized expression of a cycloidal trajectory is $\sigma(\tau) = \tau - \frac{1}{2\pi}\sin(2\pi\tau)$

Since:

$$\sigma^{'}(\tau) = 1 - \cos(2\pi\tau)$$

we have:

$$\sigma^{'}_{\max} = \sigma^{'}(0.5) = 2$$

and then:

$$\dot{s}_{\max} = 2\frac{h}{T}$$

At this point we impose the inequality:

$$2\frac{0.5}{T} < 1.5$$

which implies:

$$T > \frac{1}{1.5} = 0.66$$

4. Explain what is an artificial potential method in the context of path planning with obstacle avoidance. What is a possible issue with this method?

   In the artificial potential method, the motion of the point that represents the robot in configuration space is influenced by a potential field U. This field is obtained as the superposition of an attractive potential to the goal and a repulsive potential from the obstacles. At each configuration $\mathbf{q}$ the artificial force generated by the potential is defined as the negative gradient $-\nabla U(\mathbf{q})$ of the potential.

   A possible issue is that the method can incur in local minima.

EXERCISE 3

Consider the following system of kinematic constraints

$$\dot{q}_1 + \cos(q_1)\,\dot{q}_2 + \dot{q}_3 = 0$$
$$\sin(q_1)\,\dot{q}_1 + \dot{q}_2 - \dot{q}_3 = 0$$

where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T$.

1. Using the necessary and sufficient condition, determine if the first constraint, considered as an independent constraint, is holonomic or nonholonomic.

   Considering the first constraint

$$a^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 1 & \cos(q_1) & 1 & 0 \end{bmatrix} \dot{\mathbf{q}} = 0$$

   as an independent constraint, we can write the following equalities

$$\frac{\partial\left[\alpha(\mathbf{q})\,a_k(\mathbf{q})\right]}{\partial q_j} = \frac{\partial\left[\alpha(\mathbf{q})\,a_j(\mathbf{q})\right]}{\partial q_k}$$

for $(j,k) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, obtaining

$$\begin{aligned}
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cos\left(q_1\right)\right]}{\partial q_1} = \cos\left(q_1\right)\frac{\partial\alpha\left(\mathbf{q}\right)}{\partial q_1} - sin\left(q_1\right)\alpha\left(\mathbf{q}\right) = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_2}\\
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_1} = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_3}\\
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 0\right]}{\partial q_1} = 0 = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_4}\\
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_2} = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\cos\left(q_1\right)\right]}{\partial q_3} = \cos\left(q_1\right)\frac{\partial\alpha\left(\mathbf{q}\right)}{\partial q_3}\\
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 0\right]}{\partial q_2} = 0 = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\cos\left(q_1\right)\right]}{\partial q_4} = \cos\left(q_1\right)\frac{\partial\alpha\left(\mathbf{q}\right)}{\partial q_4}\\
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 0\right]}{\partial q_3} = 0 = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_4}
\end{aligned}$$

From the first, second and fourth equations it follows that

$$sin\left(q_1\right)\alpha\left(\mathbf{q}\right) = 0$$

and thus the only solution is $\alpha\left(\mathbf{q}\right) = 0$. As a consequence, the first is a nonholonomic constraint.

2. Using the necessary and sufficient condition, determine if the second constraint, considered as an independent constraint, is holonomic or nonholonomic.

Considering now the second constraint

$$a^T\left(\mathbf{q}\right)\dot{\mathbf{q}} = \begin{bmatrix}\sin\left(q_1\right) & 1 & -1 & 0\end{bmatrix}\dot{\mathbf{q}} = 0$$

as an independent constraint, we can write the following equalities

$$\frac{\partial\left[\alpha\left(\mathbf{q}\right)a_k\left(\mathbf{q}\right)\right]}{\partial q_j} = \frac{\partial\left[\alpha\left(\mathbf{q}\right)a_j\left(\mathbf{q}\right)\right]}{\partial q_k}$$

for $(j,k) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, obtaining

$$\begin{aligned}
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_1} = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\sin\left(q_1\right)\right]}{\partial q_2} = \sin\left(q_1\right)\frac{\partial\alpha\left(\mathbf{q}\right)}{\partial q_2}\\
&-\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_1} = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\sin\left(q_1\right)\right]}{\partial q_3} = \sin\left(q_1\right)\frac{\partial\alpha\left(\mathbf{q}\right)}{\partial q_3}\\
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 0\right]}{\partial q_1} = 0 = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\sin\left(q_1\right)\right]}{\partial q_4} = \sin\left(q_1\right)\frac{\partial\alpha\left(\mathbf{q}\right)}{\partial q_4}\\
&-\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_2} = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_3}\\
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 0\right]}{\partial q_2} = 0 = \frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_4}\\
&\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 0\right]}{\partial q_2} = 0 = -\frac{\partial\left[\alpha\left(\mathbf{q}\right)\cdot 1\right]}{\partial q_4}
\end{aligned}$$

From the first and second equations or from the fourth equation it follows that

$$\frac{\partial \alpha(\mathbf{q})}{\partial q_2} = -\frac{\partial \alpha(\mathbf{q})}{\partial q_3}$$

Instead, for example from the last equation

$$\frac{\partial \alpha(\mathbf{q})}{\partial q_4} = 0$$

As there are no other constraints, we conclude that any function $\alpha(\mathbf{q})$ that is constant with respect to q_4, and such that the derivative with respect to q_3 has an opposite value with respect to the derivative with respect to q_2 is a solution. The second constraint is thus holonomic.

3. Assuming that $A^T(\mathbf{q})\dot{\mathbf{q}} = 0$ is the Pfaffian form of the system of two constraints, and

$$g_1(\mathbf{q}) = \begin{bmatrix} 1+\cos(q_1) \\ -(1+\sin(q_1)) \\ \cos(q_1)\sin(q_1)-1 \\ 0 \end{bmatrix} \qquad g_2(\mathbf{q}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are two vectors in the null space of $A^T(\mathbf{q})$, demonstrate that the system of two constraints is holonomic.

Consider now the system of two constraints, the set can be rewritten in Pfaffian form as

$$A^T(\mathbf{q})\dot{\mathbf{q}} = \begin{bmatrix} 1 & \cos(q_1) & 1 & 0 \\ \sin(q_1) & 1 & -1 & 0 \end{bmatrix}\dot{\mathbf{q}} = \mathbf{0}$$

Taking the first and the second to last columns, it can be easily verified that rank $\left(A^T(\mathbf{q})\right) = 2$. The procedure to compute the accessibility distribution is initialized with $\Delta_1 = \text{span}\{g_1, g_2\}$.
To construct Δ_2 we have to add to Δ_1 the vector fields obtained by the Lie bracket of all possible combinations of the elements of Δ_1, that are linearly independent with respect to g_1 and g_2. The only available combination is g_1, g_2 giving rise to

$$g_3(\mathbf{q}) = [g_1, g_2] = \frac{\partial g_2}{\partial \mathbf{q}}g_1 - \frac{\partial g_1}{\partial \mathbf{q}}g_2 = \mathbf{0} - \begin{bmatrix} -\sin(q_1) & 0 & 0 & 0 \\ -\cos(q_1) & 0 & 0 & 0 \\ \cos^2(q_1)-\sin^2(q_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

As a consequence, no more vector fields can be added and $\Delta_2 = \Delta_1$.

As we have a configuration space of dimension 4, 2 constraints, and the accessibility distribution as dimension 4-2=2, we conclude that the system of constraints is holonomic.

4. Does the following kinematic model

$$\dot{\mathbf{q}} = g_1(\mathbf{q})u_1 + g_2(\mathbf{q})u_2$$

where $g_1(\mathbf{q})$ and $g_2(\mathbf{q})$ are the vectors introduced in the previous step, describe the motion of the mobile robot characterized by the system of two constraints?

We already know that $g_1(\mathbf{q})$ and $g_2(\mathbf{q})$ are two linearly independent vectors in the null space of $A^T(\mathbf{q})$. We can thus conclude that the kinematic model describes the motion of the mobile robot characterized by the system of two constraints.

EXERCISE 4

Consider the design of the trajectory tracking controller of a robot described by the rear-wheel-drive bicycle model.

1. Write the relations that allow to transform the bicycle model into the canonical simplified model. Under which assumptions do these relations hold?

   Assuming that the steering rate limit is so high that the steering angle can be changed instantaneously, we can simplify the bicycle model as

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = v\frac{\tan\phi}{\ell}$$

   where ϕ is the steering angle and ℓ the length of the bicycle.
   The relations to transform the bicycle into the canonical simplified model are

$$v = v \qquad \omega = v\frac{\tan\phi}{\ell}$$

2. Considering a point P, located along the linear velocity vector v at a distance ε from the wheel contact point, write the analytical expression of a feedback linearizing controller for the canonical simplified model.

   The equations of the feedback linearizing controller for the canonical simplified model are

$$v = v_{xP}\cos\theta + v_{yP}\sin\theta$$
$$\omega = \frac{v_{yP}\cos\theta - v_{xP}\sin\theta}{\varepsilon}$$

3. Consider the implementation of a controller as a ROS node. The following two codes (A and B) represent two different ways of implementing the node periodic loop. Which of the two is the correct implementation for a controller node? Clearly motivate the answer.

Listing 1: A

```
void controller::RunPeriodically(void)
{
    while (ros::ok())
    {
        PeriodicTask();
        ros::spinOnce();
        usleep(1000);
    }
}
```

Listing 2: B

```
void controller:RunPeriodically(float
    Period)
{
    ros::Rate LoopRate(1.0/Period);

    while (ros::ok())
    {
        PeriodicTask();
        ros::spinOnce();
        LoopRate.sleep();
    }
}
```

The correct implementation is represented by solution B.
In solution A at the end of every cycle the programme waits for a costant period of 1 ms. The duration of the while loop is not constant as the programme always wait for 1 ms independently of the time required to execute the `PeriodicTask` and the `spinOnce`.
In solution B the duration of the while loop is constant and equal to `Period`. In fact, the instruction `LoopRate.sleep()` introduces a waits that is exectly equal to the time remaining to end the loop duration.

4. Explain the difference between implementing a trajectory tracking controller where the function that computes the control variable is called in `PeriodicTask`, and one where it is called by the callback that receives the robot's position measurements. Why is it wrong to call it in the callback?

The trajectory tracking controller, as any other control system, must be executed periodically, always at the same frequency.
Calling the function that computes the control variable in the callback does not guarantee a constant execution period (e.g., in case of a delay in the measurements, or a missing measurement), and it is thus wrong. Calling the function that computes the control variable in `PeriodicTask`, instead, guarantees a constant execution period and, in case of a delay in the measurements, or a missing measurement, the callback acts as a ZOH (i.e., the controller computes the control variable on the last available measurement).