Control of Industrial and Mobile Robots

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SOLUTION

Control of Industrial and Mobile Robots
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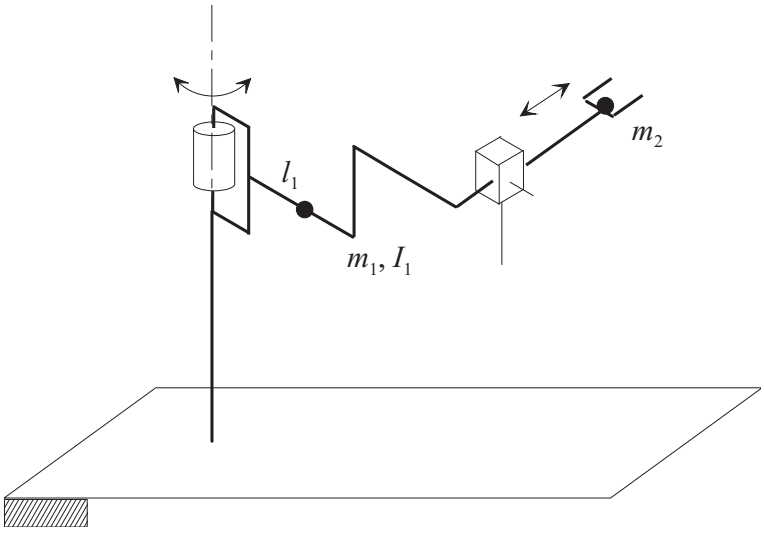
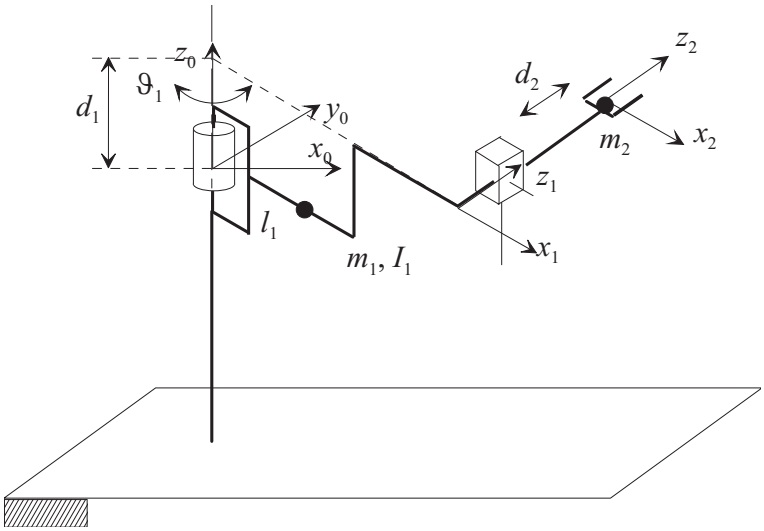
EXERCISE 1

1. Consider the manipulator sketched in the picture, where the mass of the second link is assumed to be concentrated at the end-effector:



Find the expression of the inertia matrix $\mathbf{B}(\mathbf{q})$ of the manipulator.

Denavit-Hartenberg frames can be defined as sketched in this picture:



Computations of the Jacobians:

Link 1

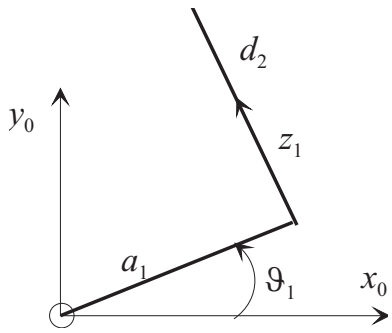
$$\mathbf{J}_P^{(l_1)} = \begin{bmatrix} \mathbf{j}_{P_1}^{(l_1)} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_{l_1} - \mathbf{p}_0) & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{J}_O^{(l_1)} = \begin{bmatrix} \mathbf{j}_{O_1}^{(l_1)} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Link 2

$$\mathbf{J}_P^{(l_2)} = \begin{bmatrix} \mathbf{j}_{P_1}^{(l_2)} & \mathbf{j}_{P_2}^{(l_2)} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_{l_2} - \mathbf{p}_0) & \mathbf{z}_1 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - d_2 c_1 & -s_1 \\ a_1 c_1 - d_2 s_1 & c_1 \\ 1 & 0 \end{bmatrix}$$

For the above computations, we can make reference to the following picture:



and to the following auxiliary vectors:

$$\mathbf{p}_{l_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ \star \end{bmatrix}, \mathbf{p}_{l_2} = \begin{bmatrix} a_1 c_1 - d_2 s_1 \\ a_1 s_1 + d_2 c_1 \\ d_1 \end{bmatrix}, \mathbf{p}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ d_1 \end{bmatrix}, \mathbf{z}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

The inertia matrix can be computed now:

$$\begin{aligned} \mathbf{B}(\mathbf{q}) &= m_1 \mathbf{J}_P^{(l_1)^T} \mathbf{J}_P^{(l_1)} + I_1 \mathbf{J}_O^{(l_1)^T} \mathbf{J}_O^{(l_1)} + m_2 \mathbf{J}_P^{(l_2)^T} \mathbf{J}_P^{(l_2)} + \\ &= m_1 \begin{bmatrix} l_1^2 & 0 \\ 0 & 0 \end{bmatrix} + I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} a_1^2 + d_2^2 & a_1 \\ a_1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \end{aligned}$$

where:

$$\begin{aligned} b_{11} &= m_1 l_1^2 + I_1 + m_2 \left(a_1^2 + d_2^2\right) \\ b_{12} &= m_2 a_1 \\ b_{22} &= m_2 \end{aligned}$$

2. Compute the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ of the Coriolis and centrifugal terms[1] for this manipulator.

   The only derivative in the Christoffel symbols which is different from zero is:

$$\frac{\partial b_{11}}{\partial q_2} = 2m_2 d_2$$

   therefore

$$\begin{array}{cc} c_{111} = 0 & c_{211} = -\frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = -m_2 d_2 \\ c_{112} = c_{121} = \frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = m_2 d_2 & c_{212} = c_{221} = 0 \\ c_{112} = 0 & c_{222} = 0 \end{array}$$

   The matrix of the Coriolis and centrifugal terms is thus:

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

   where:

$$\begin{aligned} c_{11} &= c_{111}\dot{q}_1 + c_{112}\dot{q}_2 = m_2 d_2 \dot{d}_2 \\ c_{12} &= c_{121}\dot{q}_1 + c_{122}\dot{q}_2 = m_2 d_2 \dot{\vartheta}_1 \\ c_{21} &= c_{211}\dot{q}_1 + c_{212}\dot{q}_2 = -m_2 d_2 \dot{\vartheta}_1 \\ c_{22} &= c_{221}\dot{q}_1 + c_{222}\dot{q}_2 = 0 \end{aligned}$$

3. Write the complete dynamic model for this manipulator.

   Clearly the manipulator is not affected by gravitational effects. The model is then formed by the equation:

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \tau$$

   which corresponds to the scalar equations:

$$\begin{aligned} \left(m_1 l_1^2 + I_1 + m_2\left(a_1^2 + d_2^2\right)\right)\ddot{\vartheta}_1 + m_2 a_1 \ddot{d}_2 + 2m_2 d_2 \dot{\vartheta}_1 \dot{d}_2 &= \tau_1 \\ m_2 a_1 \ddot{\vartheta}_1 + m_2 \ddot{d}_2 - m_2 d_2 \dot{\vartheta}_1^2 &= \tau_2 \end{aligned}$$

4. Show that the model obtained in the previous step is linear with respect to a set of dynamic parameters.

   The model can be written in the form:

[1] The general expression of the Christoffel symbols is $c_{ijk} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)$

$$\mathbf{Y}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right) \mathbf{\Pi} = \tau$$

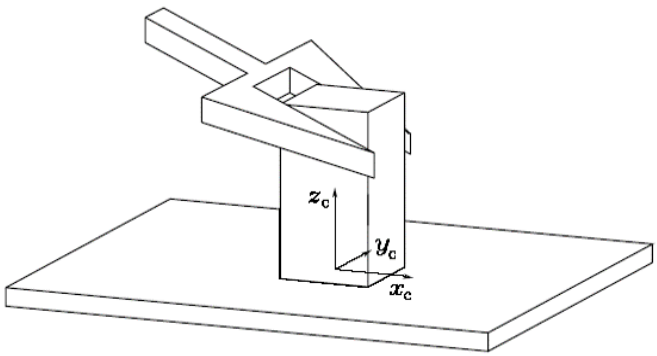
with:

$$\mathbf{\Pi} = \begin{bmatrix} m_1 l_1^2 + I_1 \\ m_2 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} \ddot{\vartheta}_1 & \left(a_1^2 + d_2^2\right) \ddot{\vartheta}_1 + a_1 \ddot{d}_2 + 2 d_2 \dot{\vartheta}_1 \dot{d}_2 \\ 0 & a_1 \ddot{\vartheta}_1 + \ddot{d}_2 - d_2 \dot{\vartheta}_1^2 \end{bmatrix}$$

EXERCISE 2

1. Consider an interaction task of a manipulator, with a frictionless and rigid surface, as in this picture:



Express the natural and the artificial constraints for this problem, and specify the selection matrix.

The natural constraints and artificial constraints can be easily identified:

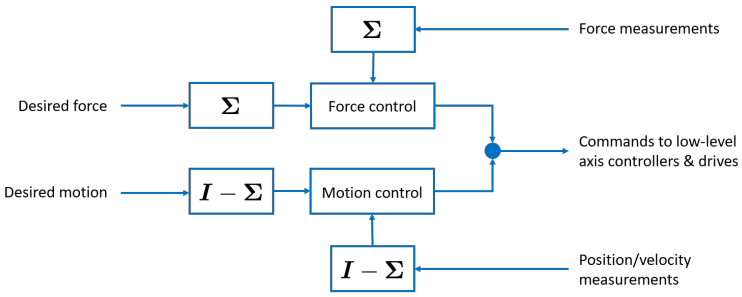
Natural constraints	Artificial constraints
f_x^c	$\dot{p}_x^c$
f_y^c	$\dot{p}_y^c$
$\dot{p}_z^c$	f_z^c
ω_x^c	μ_x^c
ω_y^c	μ_y^c
μ_z^c	ω_z^c

The selection matrix is thus:

$$\Sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Sketch the block diagram of a hybrid force-position controller. What are possible sources of inconsistency in the adoption of such scheme?

   The block diagram of a hybrid force-position controller is as follows:



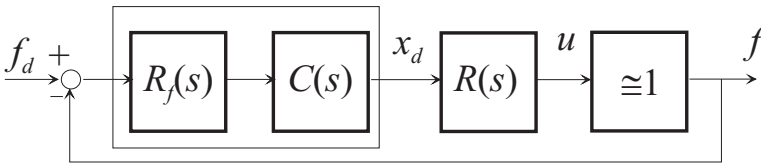
   Possible sources of inconsistency are friction at the contact (a force is detected in a nominally free direction), compliance in the robot structure and/or at the contact (a displacement is detected in a direction which is nominally constrained in motion), uncertainty in the environment geometry at the contact.

3. Explain what an implicit force controller is and why it might be convenient with respect to an explicit solution.

   An implicit force control is closed around the position control loops. This is usually the only viable solution to implement force control, since the reliable and industrially safe position controllers cannot be bypassed.

4. Suppose now that along the translational z direction an implicit force controller has to be designed. Sketch the block diagram of such controller and design it taking a bandwidth of 20 rad/s.

   The block diagram of an implicit force controller in case of rigid surface is sketched in the picture:



   where $R(s)$ is the transfer function of the position controller. If we assume a PID position controller:

$$R(s) = \frac{K_D s^2 + K_P s + K_I}{s}$$

   The partial compensator of such controller is:

$$C(s) = \frac{1}{K_D s^2 + K_P s + K_I}$$

If we select a PI controller on the force error:

$$R_f(s) = k_{pf} + \frac{k_{if}}{s}$$

the loop transfer function becomes:

$$L_f(s) = \frac{sk_{pf} + k_{if}}{s^2}$$

Since the high frequency approximation of such transfer function is k_{pf}/s we can set $k_{pf} = 20$ (equal to the required bandwidth. The zero of the controller can be set at a lower frequency range, for example $k_{if}/k_{pf} = 2$, which yields $k_{if} = 40$.

EXERCISE 3

1. Consider a unicycle mobile robot. Selecting as flat outputs $z_1 = x$ and $z_2 = y$, write the flat model of the robot, i.e., the analytical relations from z_1, z_2 to x, y, θ and from z_1, z_2 to v, ω.

   The flatness transformation for the state is given by

$$x = z_1 \qquad y = z_2 \qquad \theta = \begin{cases} \arctan\left(\frac{\dot{z}_2}{\dot{z}_1}\right) & \dot{z}_1 > 0 \\ \pi + \arctan\left(\frac{\dot{z}_2}{\dot{z}_1}\right) & \dot{z}_1 < 0 \\ \frac{\pi}{2}\text{sign}\left(\dot{z}_2\right) & \dot{z}_1 = 0 \end{cases}$$

   and for the input

$$v = \sqrt{\dot{z}_1 + \dot{z}_2} \qquad \omega = \frac{\dot{z}_1\ddot{z}_2 - \ddot{z}_1\dot{z}_2}{\dot{z}_1 + \dot{z}_2}$$

2. Using the flatness transformation, determine the analytic expression of a trajectory $x(t)$, $y(t)$ (and the numerical values of its coefficients) that moves a unicycle robot, in an obstacle free environment, from an initial state $x_i = y_i = \theta_i = 0$ and $v_i = 0$ at $t_i = 0$, to a final state $x_f = y_f = 5$, $\theta_f = 0$ and $v_f = 0$ at $t_f = 1$.

   Considering that we have 4 initial and 4 final conditions we can select two third order polynomials for z_1 and z_2, as follows

$$z_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \qquad z_2(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

   whose first order derivatives are

$$\dot{z}_1(t) = a_1 + 2a_2 t + 3a_3 t^2 \qquad \dot{z}_2(t) = b_1 + 2b_2 t + 3b_3 t^2$$

   Imposing the initial position and the initial velocity we get

$$a_0 = b_0 = 0 \qquad a_1 = b_1 = 0$$

   Imposing now the final position and the final velocity we get

$$a_2 + a_3 = 5 \qquad b_2 + b_3 = 5 \qquad 2a_2 + 3a_3 = 0 \qquad 2b_2 + 3b_3 = 0$$

and solving the two systems of linear equations

$$a_2 = b_2 = 15 \qquad a_3 = b_3 = -10$$

The resulting trajectory is

$$x(t) = 15t^2 - 10t^3 \qquad y(t) = 15t^2 - 10t^3$$

3. Modify the answer to the previous step in order to introduce the minimization of the cost

$$J(v, \omega) = \int_0^{T_f} \left(v^2 + 0.1\omega^2\right) \, \mathrm{d}t$$

where now T_f is a free parameter. Write the analytical expression of the relations that allow to compute the additional coefficients that must be introduced in order to enforce the minimization of the cost function.

We can increase the order of the polynomial representing z_1, obtaining

$$z_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \qquad z_2(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

We still have $a_0 = b_0 = 0$ and $a_1 = b_1 = 0$, consequently

$$z_1(t) = a_2 t^2 + a_3 t^3 + a_4 t^4 \qquad z_2(t) = b_2 t^2 + b_3 t^3$$

and the derivatives with respect to time are

$$\dot{z}_1(t) = 2a_2 t + 3a_3 t^2 + 4a_4 t^3 \qquad \dot{z}_2(t) = 2b_2 t + 3b_3 t^2$$

Imposing now the final position and the final velocity we get

$$a_2 T_f^2 + a_3 T_f^3 + a_4 = 5 \qquad b_2 T_f^2 + b_3 T_f^3 = 5 \qquad 2a_2 T_f + 3a_3 T_f^2 + 4a_4 T_f^3 = 0 \qquad 2b_2 T_f + 3b_3 T_f^2 = 0$$

We can represent these four equations in the following linear system

$$\begin{bmatrix} T_f^2 & T_f^3 & 0 & 0 \\ 2T_f & 3T_f^2 & 0 & 0 \\ 0 & 0 & T_f^2 & T_f^3 \\ 0 & 0 & 2T_f & 3T_f^2 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 5 - a_4 \\ -4T_f^3 a_4 \\ 5 \\ 0 \end{bmatrix}$$

that can be solved obtaining a_2, a_3, b_2, b_3 as functions of a_4 and T_f.
Finally, a_4 and T_f can be computed enforcing the minimization of the cost function.

4. Consider now an environment with obstacles, where each obstacle can be represented by a circle of radius R_i and center (c_{x_i}, c_{y_i}). Write the constraint that must be included in the optimization problem considered in the previous step, in order to guarantee obstacle avoidance.

For each obstacle i of radius R_i and center (c_{x_i}, c_{y_i}), one has to include a constraint

$$(x - c_{x_i})^2 + (y - c_{y_i})^2 \geq R_i^2$$

EXERCISE 4

Consider the design of a trajectory tracking controller for a unicycle robot based on feedback linearization.

1. Write the analytical relations that define the coordinate transformation from the unicycle wheel contact point to point P, i.e., the new reference point considered to solve the trajectory tracking problem.

   The analytical relations that define the coordinate transformation from the unicycle wheel contact point to point P are given by

$$x_P = x + \varepsilon \cos\theta$$
$$y_P = y + \varepsilon \sin\theta$$

   where ε is the distance of point P from the unicycle wheel contact point (x, y), and θ is the robot orientation.

2. Starting from the coordinate transformation in step 1, derive the control laws of the feedback linearizing controller.

   Taking the derivative with respect to time of the coordinate transformation we obtain

$$\dot{x}_P = \dot{x} - \varepsilon\dot{\theta}\sin\theta = v\cos\theta - \varepsilon\omega\sin\theta$$
$$\dot{y}_P = \dot{y} + \varepsilon\dot{\theta}\cos\theta = v\sin\theta + \varepsilon\omega\cos\theta$$

   that can be rewritten in matrix form as

$$\begin{bmatrix} \dot{x}_P \\ \dot{y}_P \end{bmatrix} = \begin{bmatrix} \cos\theta & -\varepsilon\sin\theta \\ \sin\theta & \varepsilon\cos\theta \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

   Defining $v_{P_x} = \dot{x}_P$, $v_{P_y} = \dot{y}_P$ and inverting the relation we obtain the analytical expression of the feedback linearizing controller

$$v = v_{P_x}\cos\theta + v_{P_y}\sin\theta$$
$$\omega = \frac{v_{P_y}\cos\theta - v_{P_x}\sin\theta}{\varepsilon}$$

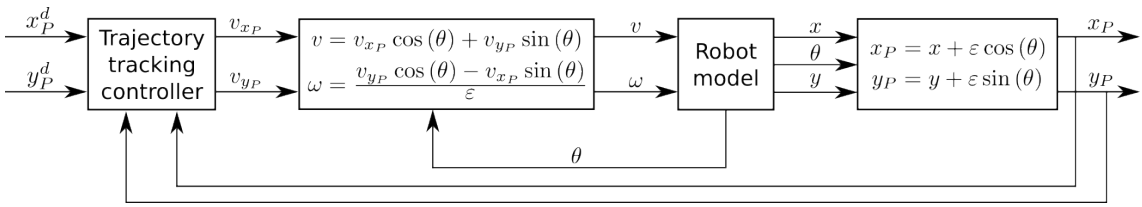
3. Using the control laws derived in the previous step and the unicycle kinematic model, derive the expression of the dynamic system representing the closed-loop system obtained connecting the linearizing controller and the kinematic model.

   The dynamic model representing the closed-loop system has the following expression

$$\dot{x} = v_{P_x}\cos^2\theta + v_{P_y}\sin\theta\cos\theta$$
$$\dot{y} = v_{P_x}\cos\theta\sin\theta + v_{P_y}\sin^2\theta$$
$$\dot{\theta} = \frac{v_{P_y}\cos\theta - v_{P_x}\sin\theta}{\varepsilon}$$

4. Draw the block diagram of the complete trajectory tracking controller, including the feedback linearizing controller, the robot model, and the trajectory tracking controller. Write the equations of the dynamic system that must be used in order to design the trajectory tracking controller.

The block diagram of the complete trajectory tracking controller is shown in the figure below.



The dynamic system that must be used in order to design the trajectory tracking controller is given by

$$\dot{x}_P = v_{P_x}$$
$$\dot{y}_P = v_{P_y}$$