# Control of Industrial and Mobile Robots 

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Jandary 11, 2023

## SOLUTION

# Control of Industrial and Mobile Robots Prof. Luca Bascetta and Prof. Paolo Rocco 

## EXERCISE 1

Consider the manipulator sketched in the picture:


1. Find the expression of the inertia matrix $\mathbf{B}(\mathbf{q})$ of the manipulator ${ }^{1}$

Denavit-Hartenberg frames can be defined as sketched in this picture:


Computations of the Jacobians:
${ }^{1}$ The cross product between vector $a=\left[\begin{array}{l}a_{1} \\ a_{2} \\ a_{3}\end{array}\right]$ and $b=\left[\begin{array}{c}b_{1} \\ b_{2} \\ b_{3}\end{array}\right]$ is $c=a \times b=\left[\begin{array}{l}a_{2} b_{3}-a_{3} b_{2} \\ a_{3} b_{1}-a_{1} b_{3} \\ a_{1} b_{2}-a_{2} b_{1}\end{array}\right]$

Link 1

$$
\mathbf{J}_{P}^{\left(l_{1}\right)}=\left[\begin{array}{cc}
\mathbf{j}_{P_{1}}^{\left(l_{1}\right)} & \mathbf{0}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{z}_{0} & \mathbf{0}
\end{array}\right]=\left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
1 & 0
\end{array}\right]
$$

Link 2

$$
\begin{gathered}
\mathbf{J}_{P}^{\left(l_{2}\right)}=\left[\begin{array}{ll}
\mathbf{j}_{P_{1}}^{\left(l_{2}\right)} & \mathbf{j}_{P_{2}}^{\left(l_{2}\right)}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{z}_{0} & \mathbf{z}_{1} \times\left(\mathbf{p}_{l_{2}}-\mathbf{p}_{1}\right)
\end{array}\right]=\left[\begin{array}{cc}
0 & -l_{2} s_{2} \\
0 & 0 \\
1 & l_{2} c_{2}
\end{array}\right] \\
\mathbf{J}_{O}^{\left(l_{2}\right)}=\left[\begin{array}{ll}
\mathbf{j}_{O_{1}}^{\left(l_{2}\right)} & \mathbf{j}_{O_{2}}^{\left(l_{2}\right)}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{0} & \mathbf{z}_{1}
\end{array}\right]=\left[\begin{array}{cc}
0 & 0 \\
0 & -1 \\
0 & 0
\end{array}\right]
\end{gathered}
$$

For the above computations, we can make reference to the following picture:

and to the following auxiliary vectors:

$$
\mathbf{p}_{l_{2}}=\left[\begin{array}{c}
-a_{1}+l_{2} c_{2} \\
0 \\
d_{1}+l_{2} s_{2}
\end{array}\right], \mathbf{p}_{1}=\left[\begin{array}{c}
-a_{1} \\
0 \\
d_{1}
\end{array}\right], \mathbf{z}_{1}=\left[\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right]
$$

The inertia matrix can be computed now:

$$
\begin{aligned}
\mathbf{B}(\mathbf{q}) & =m_{1} \mathbf{J}_{P}^{\left(l_{1}\right)^{T}} \mathbf{J}_{P}^{\left(l_{1}\right)}+m_{2} \mathbf{J}_{P}^{\left(l_{2}\right)^{T}} \mathbf{J}_{P}^{\left(l_{2}\right)}+I_{2} \mathbf{J}_{O}^{\left(l_{2}\right)^{T}} \mathbf{J}_{O}^{\left(l_{2}\right)} \\
& =m_{1}\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]+m_{2}\left[\begin{array}{cc}
1 & l_{2} c_{2} \\
l_{2} c_{2} & l_{2}^{2}
\end{array}\right]+I_{2}\left[\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right] \\
& =\left[\begin{array}{ll}
b_{11} & b_{12} \\
b_{12} & b_{22}
\end{array}\right]
\end{aligned}
$$

where:

$$
\begin{aligned}
b_{11} & =m_{1}+m_{2} \\
b_{12} & =m_{2} l_{2} c_{2} \\
b_{22} & =m_{2} l_{2}^{2}+I_{2}
\end{aligned}
$$

2. Compute the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ of the Coriolis and centrifugal terms ${ }^{2}$ for this manipulator.

The only derivative in the Christoffel symbols which is different from zero is:

$$
\frac{\partial b_{12}}{\partial q_{2}}=\frac{\partial b_{21}}{\partial q_{2}}=-m_{2} l_{2} s_{2}
$$

therefore

$$
\begin{array}{cc}
c_{111}=0 & c_{211}=0 \\
c_{112}=c_{121}=0 & c_{212}=c_{221}=\frac{1}{2}\left(\frac{\partial b_{21}}{\partial q_{2}}-\frac{\partial b_{12}}{\partial q_{2}}\right)=0 \\
c_{122}=\frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_{2}}+\frac{\partial b_{21}}{\partial q_{2}}\right)=-m_{2} l_{2} s_{2} & c_{222}=0
\end{array}
$$

The matrix of the Coriolis and centrifugal terms is thus:

$$
\mathbf{C}=\left[\begin{array}{ll}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{array}\right]
$$

where:

$$
\begin{aligned}
& c_{11}=c_{111} \dot{q}_{1}+c_{112} \dot{q}_{2}=0 \\
& c_{12}=c_{121} \dot{q}_{1}+c_{122} \dot{q}_{2}=-m_{2} l_{2} s_{2} \dot{\vartheta}_{2} \\
& c_{21}=c_{211} \dot{q}_{1}+c_{212} \dot{q}_{2}=0 \\
& c_{22}=c_{221} \dot{q}_{1}+c_{222} \dot{q}_{2}=0
\end{aligned}
$$

3. Check that matrix $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})=\dot{\mathbf{B}}(\mathbf{q})-2 \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew symmetric.

We have that:

$$
\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})=\dot{\mathbf{B}}(\mathbf{q})-2 \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})=\left[\begin{array}{cc}
0 & -m_{2} l_{2} s_{2} \dot{\vartheta}_{2} \\
-m_{2} l_{2} s_{2} \dot{\vartheta}_{2} & 0
\end{array}\right]-2\left[\begin{array}{cc}
0 & -m_{2} l_{2} s_{2} \dot{\vartheta}_{2} \\
0 & 0
\end{array}\right]=\left[\begin{array}{cc}
0 & m_{2} l_{2} s_{2} \dot{\vartheta}_{2} \\
-m_{2} l_{2} s_{2} \dot{\vartheta}_{2} & 0
\end{array}\right]
$$

which is a skew-symmetric matrix.
4. For a generic manipulator, ignoring the gravitational terms and exploiting the skew symmetry of matrix $\mathbf{N}$, obtain an expression of the derivative with respect to time of the kinetic energy.

The kinetic energy is:

$$
T=\frac{1}{2} \dot{\mathbf{q}}^{T} \mathbf{B}(\mathbf{q}) \dot{\mathbf{q}}
$$

[^0]Its time derivative is:

$$
\frac{d T}{d t}=\dot{\mathbf{q}}^{T} \mathbf{B}(\mathbf{q}) \ddot{\mathbf{q}}+\frac{1}{2} \dot{\mathbf{q}}^{T} \dot{\mathbf{B}}(\mathbf{q}) \dot{\mathbf{q}}
$$

Exploiting the dynamic model we have:

$$
\frac{d T}{d t}=\dot{\mathbf{q}}^{T}[-\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})+\tau]+\frac{1}{2} \dot{\mathbf{q}}^{T} \dot{\mathbf{B}}(\mathbf{q}) \dot{\mathbf{q}}=\frac{1}{2} \dot{\mathbf{q}}^{T}[\dot{\mathbf{B}}-2 \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})] \dot{\mathbf{q}}+\dot{\mathbf{q}}^{T} \tau
$$

Thanks to the skew symmetry of the matrix, we finally obtain:

$$
\frac{d T}{d t}=\dot{\mathbf{q}}^{T} \tau
$$

EXERCISE 2 Consider a robot that uses a camera.

1. Explain what are the extrinsic and the intrinsic calibrations, making in particular reference to the notion of camera intrinsic matrix.

The extrinsic calibration is the determination of the extrinsic parameters of the camera, like the position and the orientation of the camera with respect to a reference frame. The intrinsic calibration is the determination of the intrinsic parameters of the camera (like the focal length $\lambda$ ) as well as of some additional parameters. The intrinsic parameters are usually organized in a matrix (camera intrinsic matrix):

$$
\mathbf{K}=\left[\begin{array}{ccc}
f_{x} & s & c_{x} \\
0 & f_{y} & c_{y} \\
0 & 0 & 1
\end{array}\right]
$$

where $c_{x}$ and $c_{y}$ are the coordinates of the optical center, $f_{x}$ and $f_{y}$ are the ratios between the focal length and the size (along x and y ) of the pixel, $s$ is a skew parameter.
2. With reference to the following sketch, define what an image feature is and write the equations of the perspective projection method.


The image feature is the coding of any information that can be retrieved from an image, for example the two coordinates of a point in the image plane. The equations of the perspective projection can be written as:

$$
\xi=\left[\begin{array}{l}
u \\
v
\end{array}\right]=\frac{\lambda}{Z}\left[\begin{array}{l}
X \\
Y
\end{array}\right]
$$

3. Define the interaction matrix and the image Jacobian for a vision-based robotic system, in terms of the quantities that each of the two matrices relate.

The interaction matrix relates the linear and angular velocities of the camera to the velocity in the image plane:

$$
\left[\begin{array}{c}
\dot{u} \\
\dot{v}
\end{array}\right]=\mathbf{L}\left[\begin{array}{c}
\dot{\mathbf{O}}_{c} \\
\omega_{c}
\end{array}\right]
$$

The image Jacobian relates the joint velocities of the robot to the velocity in the image plane:

$$
\left[\begin{array}{c}
\dot{u} \\
\dot{v}
\end{array}\right]=\mathbf{J}_{I} \dot{\mathbf{q}}
$$

4. Consider now the following block diagram:


Is this a look-and-move or a visual servoing scheme? A position-based or an image-based scheme? Write an expression of the control law that can be used in this control scheme.

The scheme corresponds to a look-and-move image-based control scheme. The control law can be written as:

$$
\dot{\mathbf{q}}=\mathbf{J}_{I}^{\sharp}\left(\dot{\xi}_{d}+K\left(\xi_{d}-\xi\right)\right)+\left(\mathbf{I}-\mathbf{J}_{I}^{\sharp} \mathbf{J}_{I}\right) \dot{\mathbf{q}}_{0}
$$

## EXERCISE 3

1. Given the kinematic constraint

$$
\dot{q}_{1}+\dot{q}_{4}=0
$$

where $\mathbf{q} \in \mathbb{R}^{4}$ is the configuration vector. Determine, using the necessary and sufficient condition, if this constraint is holonomic or nonholonomic.

It is straightforward that this constraint is holonomic, as it is equivalent to $q_{1}(t)+q_{4}(t)=c$.
Applying the necessary and sufficient condition

$$
\begin{aligned}
\frac{\partial \alpha(\mathbf{q})}{\partial q_{2}} & =0 \\
\frac{\partial \alpha(\mathbf{q})}{\partial q_{3}} & =0 \\
\frac{\partial \alpha(\mathbf{q})}{\partial q_{4}} & =\frac{\partial \alpha(\mathbf{q})}{\partial q_{1}} \\
0 & =0 \\
0 & =\frac{\partial \alpha(\mathbf{q})}{\partial q_{2}} \\
0 & =\frac{\partial \alpha(\mathbf{q})}{\partial q_{3}}
\end{aligned}
$$

Any function $\alpha(\mathbf{q})$ that satisfies the first three equations is a solution of this system, e.g., $\alpha(\mathbf{q})=$ $q_{1}+q_{4}$. The constraint is thus holonomic.
2. Given the kinematic constraint

$$
q_{1} \dot{q}_{2}+\dot{q}_{3}=0
$$

where $\mathbf{q} \in \mathbb{R}^{4}$ is the configuration vector. Determine, using the necessary and sufficient condition, if this constraint is holonomic or nonholonomic.

Applying the necessary and sufficient condition we obtain

$$
\begin{aligned}
& 0=\frac{\partial\left(\alpha(\mathbf{q}) q_{1}\right)}{\partial q_{1}}=q_{1} \frac{\partial \alpha(\mathbf{q})}{\partial q_{1}}+\alpha(\mathbf{q}) \\
& 0=\frac{\partial \alpha(\mathbf{q})}{\partial q_{1}}
\end{aligned}
$$

The first two conditions are enough to conclude $\alpha(\mathbf{q})=0$, and the constraint is thus nonholonomic.
3. Is the system of constraints

$$
\begin{aligned}
\dot{q}_{1}+\dot{q}_{4} & =0 \\
q_{1} \dot{q}_{2}+\dot{q}_{3} & =0
\end{aligned}
$$

holonomic or nonholonomic? Motivate the answer analysing the accessibility distribution.
Note that $\operatorname{rank}\left(A^{T}(\mathbf{q})\right)=2$, and two vectors in the null space of $A^{T}(\mathbf{q})$ are

$$
g_{1}(\mathbf{q})=\left[\begin{array}{c}
0 \\
1 \\
-q_{1} \\
0
\end{array}\right] \quad g_{2}(\mathbf{q})=\left[\begin{array}{c}
1 \\
0 \\
0 \\
-1
\end{array}\right]
$$

The procedure to compute the accessibility distribution is initialized with $\Delta_{1}=\operatorname{span}\left\{g_{1}, g_{2}\right\}$. To construct $\Delta_{2}$ we have to add to $\Delta_{1}$ the vector fields obtained by the Lie bracket of all possible combinations of the elements of $\Delta_{1}$, that are linearly independent with respect to $g_{1}$ and $g_{2}$. The only available combination is $g_{1}, g_{2}$ giving rise to

$$
g_{3}(\mathbf{q})=\left[g_{1}, g_{2}\right]=\frac{\partial g_{2}}{\partial \mathbf{q}} g_{1}-\frac{\partial g_{1}}{\partial \mathbf{q}} g_{2}=\mathbf{0}-\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
1 \\
0 \\
0 \\
-1
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right]
$$

It is straightforward to check that $g_{1}, g_{2}$, and $g_{3}$ are linearly independent, as a consequence $\Delta_{2}=$ $\operatorname{span}\left\{g_{1}, g_{2}, g_{3}\right\}$.
Again to construct $\Delta_{3}$ we have to add to $\Delta_{2}$ the vector fields obtained by the Lie bracket of all possible combinations including one element of $\Delta_{2}$ and one of $\Delta_{1}$, that are linearly independent with respect to $g_{1}, g_{2}$ and $g_{3}$. The available candidates are (already excluding Brackets that are equal to 0 , e.g., $\left[g_{1}, g_{1}\right]$, and Brackets that are equal except for the sign, e.g. $\left[g_{1}, g_{2}\right]$ and $\left[g_{2}, g_{1}\right]$ )

$$
\left[g_{1}, g_{2}\right] \quad\left[g_{1}, g_{3}\right] \quad\left[g_{2}, g_{3}\right]
$$

Excluding the first one that is already in $\Delta_{2}$ as $g_{3}$, we can compute the second and the third

$$
\begin{aligned}
& {\left[g_{1}, g_{3}\right]=\frac{\partial g_{3}}{\partial \mathbf{q}} g_{1}-\frac{\partial g_{1}}{\partial \mathbf{q}} g_{3}=\mathbf{0}-\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right]=\mathbf{0}} \\
& {\left[g_{2}, g_{3}\right]=\frac{\partial g_{3}}{\partial \mathbf{q}} g_{2}-\frac{\partial g_{2}}{\partial \mathbf{q}} g_{3}=\mathbf{0}}
\end{aligned}
$$

No more vectors can be added, the accessibility distribution is thus $\Delta_{A}=\Delta_{2}=\operatorname{span}\left\{g_{1}, g_{2}, g_{3}\right\}$. As a consequence, the accessibility space has dimension 3 , that is equal to $n-k$, and thus the system of constraints is holonomic.
4. Consider a mobile robot, whose configuration is represented by $\mathbf{q} \in \mathbb{R}^{4}$, and whose motion is described by the set of constraints introduced in the previous question.
Does the following kinematic model

$$
\dot{\mathbf{q}}=\left[\begin{array}{c}
0 \\
1 \\
-q_{1} \\
0
\end{array}\right] u_{1}+\left[\begin{array}{c}
1 \\
0 \\
0 \\
-1
\end{array}\right] u_{2}
$$

describe the motion of the robot? Clearly motivate the answer.

The two vectors

$$
\left[\begin{array}{c}
0 \\
1 \\
-q_{1} \\
0
\end{array}\right] \quad\left[\begin{array}{c}
1 \\
0 \\
0 \\
-1
\end{array}\right]
$$

are two independent vectors in the null space of $A^{T}(\mathbf{q})$, the given kinematic model thus describes the motion of the robot.

## EXERCISE 4

1. The trajectory tracking controller for a robot, modelled using the bicycle kinematic model, is designed exploiting the canonical simplified model. Write the relation that allows to transform the bicycle kinematic model into the canonical simplified model.
Under which assumptions the transformation can be applied?

Assuming that the steering rate is so high that the steering angle can be changed instantaneously, we can simplify the bicycle model reducing it to only three states. Under this assumption, applying the following relations

$$
v=v \quad \omega=v \frac{\tan \phi}{\ell}
$$

the bicycle model is transformed into the canonical simplified model.
2. The robot is characterised by the following actuation constraints $0 \leq v \leq v_{M}$ and $\phi_{m} \leq \phi \leq \phi_{M}$. Show how these constraints can be remapped into constraints on the control variables of the canonical simplified model.

The first constraint, i.e., $0 \leq v \leq v_{M}$, is already expressed with respect to a control variable of the canonical simplified model, consequently no remapping is needed.
Consider now the second constraint, i.e., $\phi_{m} \leq \phi \leq \phi_{M}$, thanks to the fact that the tangent function is monotonically increasing, it can be rewritten as

$$
\tan \left(\phi_{m}\right) \leq \tan (\phi) \leq \tan \left(\phi_{M}\right)
$$

Substituting now the relation between $\tan (\phi)$ and $\omega$ we obtain

$$
\tan \left(\phi_{m}\right) \leq \frac{\omega \ell}{v} \leq \tan \left(\phi_{M}\right)
$$

and considering that the velocity is positive

$$
\frac{v}{\ell} \tan \left(\phi_{m}\right) \leq \omega \leq \frac{v}{\ell} \tan \left(\phi_{M}\right)
$$

3. Draw the block diagram of the complete trajectory tracking controller based on the canonical simplified model transformation, and write the equations representing each block.

The block diagram is reported in Figure 1.


Figure 1: Trajectory tracking controller block diagram.
The inverse canonical model transformation is described by the following relations

$$
v=v \quad \phi=\arctan \left(\frac{\omega \ell}{v}\right)
$$

The feedback linearisation controller is described by the following relations

$$
v=v_{x_{P}} \cos \theta+v_{y_{P}} \sin \theta \quad \omega=\frac{v_{y_{P}} \cos \theta-v_{x_{P}} \sin \theta}{\varepsilon}
$$

The point $P$ transformation is described by the following relations

$$
x_{P}=x+\varepsilon \cos \theta \quad y_{P}=y+\varepsilon \sin \theta
$$

The trajectory tracking controller is composed of two independent P or PI regulators.
4. A more simple control solution can be devised transforming the bicycle into the canonical simplified model, and then adopting an open-loop control strategy based on the flatness transformation. Write (and explain) the equations of the trajectory tracking controller, assuming that an analytical expression of the desired trajectory, as two functions of time, $x^{d}(t), y^{d}(t)$, is available to the controller.

If an analytical expression of the desired trajectory is available to the controller, we can assume that all the required derivatives with respect to time are available as well.
The controller can be thus based on the flatness transformation of the unicycle kinematic model, mapping the desired trajectory into a desired value of the control variables $v$ and $\omega$. The inverse canonical model transformation is then used to compute $v$ and $\phi$.
The equations of the controller are the following

$$
\begin{aligned}
& v=\sqrt{\dot{x}^{d^{2}}+\dot{y}^{d^{2}}} \\
& \omega=\frac{\dot{x}^{d} \dot{y}^{d}-\dot{y}^{d} \ddot{x}^{d}}{\dot{x}^{d^{2}}+\dot{y}^{d^{2}}} \\
& \phi=\arctan \left(\frac{\omega \ell}{v}\right)
\end{aligned}
$$

where $\dot{x}^{d}, \ddot{x}^{d}, \dot{y}^{d}, \ddot{y}^{d}$ are the inputs, and $v, \phi$ the outputs of the controller.


[^0]:    ${ }^{2}$ The general expression of the Christoffel symbols is $c_{i j k}=\frac{1}{2}\left(\frac{\partial b_{i j}}{\partial q_{k}}+\frac{\partial b_{i k}}{\partial q_{j}}-\frac{\partial b_{j k}}{\partial q_{i}}\right)$

