

# Control of industrial robots

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February 3, 2015

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## Warnings:

- This file consists of **8** pages (including cover). All the pages should be signed.
- During the exam you are not allowed to exit the room for any other reason than handing your work or withdrawing from the exam.
- You are not allowed to withdraw from the exam during the first 30 minutes.
- During the exam you are not allowed to consult books or any kind of notes.
- You are not allowed to use calculators with graphic display.
- Solutions and answers can be given **either in English or in Italian**.
- Solutions and answers must be given **exclusively in the reserved space**. Only in the case of corrections, or if the space is not sufficient, use the back of the front cover.
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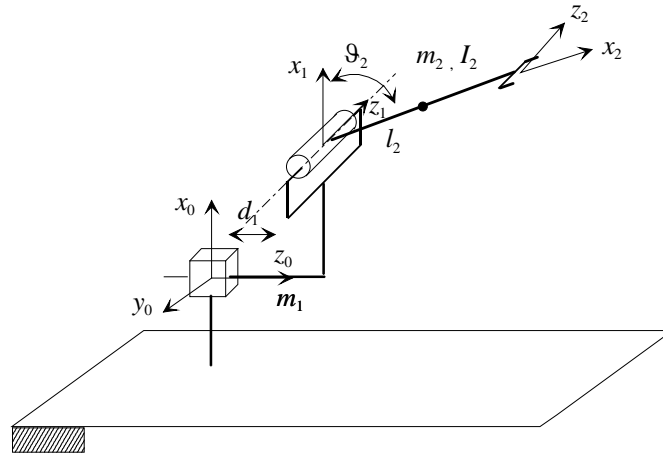
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**Use this page ONLY in case of corrections or if the space reserved for some answers turned out to be insufficient**

**Exercise 1**

Consider the manipulator sketched in the picture:

**1.1** Find the expression of the inertia matrix of the manipulator<sup>1</sup>.

Denavit-Hartenberg frames can be defined as sketched in the picture.

Computations of the Jacobians:

Link 1

$$J_P^{(l_1)} = [j_{P1}^{(l_1)} \quad \mathbf{0}] = [z_0 \quad \mathbf{0}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

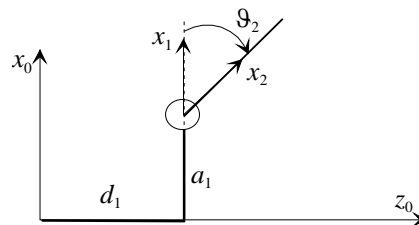
Link 2

$$J_P^{(l_2)} = [j_{P1}^{(l_2)} \quad j_{P2}^{(l_2)}] = [z_0 \quad z_1 \times (p_{l_2} - p_1)] = \begin{bmatrix} 0 & -l_2 s_2 \\ 0 & 0 \\ 1 & l_2 c_2 \end{bmatrix}$$

$$J_O^{(l_2)} = [j_{O1}^{(l_2)} \quad j_{O2}^{(l_2)}] = [\mathbf{0} \quad z_1] = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Auxiliary vectors for the above computations:

$$p_{l_2} = \begin{bmatrix} a_1 + l_2 c_2 \\ 0 \\ d_1 + l_2 s_2 \end{bmatrix}, \quad p_1 = \begin{bmatrix} a_1 \\ 0 \\ d_1 \end{bmatrix}, \quad z_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$



<sup>1</sup> The cross product between vectors  $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$  and  $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$  is  $c = a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$

Inertia matrix:

$$\mathbf{B}(\mathbf{q}) = m_1 \mathbf{J}_P^{(l_1)T} \mathbf{J}_P^{(l_1)} + m_2 \mathbf{J}_P^{(l_2)T} \mathbf{J}_P^{(l_2)} + I_2 \mathbf{J}_O^{(l_2)T} \mathbf{J}_O^{(l_2)} = m_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} 1 & l_2 c_2 \\ l_2 c_2 & l_2^2 \end{bmatrix} + I_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

with:

$$b_{11} = m_1 + m_2$$

$$b_{12} = m_2 l_2 c_2$$

$$b_{22} = m_2 l_2^2 + I_2$$

**1.2** Compute the matrix  $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$  of the Coriolis and centrifugal terms <sup>2</sup> for this manipulator.

The only derivative in the Christoffel symbols which is different from zero is:

$$\frac{\partial b_{12}}{\partial q_2} = \frac{\partial b_{21}}{\partial q_2} = -m_2 l_2 c_2$$

Therefore:

$$c_{111} = 0 \quad c_{211} = 0$$

$$c_{112} = c_{121} = 0 \quad c_{212} = c_{221} = \frac{1}{2} \left( \frac{\partial b_{12}}{\partial q_2} - \frac{\partial b_{12}}{\partial q_2} \right) = 0$$

$$c_{122} = \frac{1}{2} \left( \frac{\partial b_{12}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_2} \right) = -m_2 l_2 s_2 \quad c_{222} = 0$$

The matrix of the Coriolis and centrifugal terms is thus:

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad \begin{aligned} c_{11} &= c_{111} \dot{q}_1 + c_{112} \dot{q}_2 = 0 \\ c_{12} &= c_{121} \dot{q}_1 + c_{122} \dot{q}_2 = -m_2 l_2 s_2 \dot{q}_2 \\ c_{21} &= c_{211} \dot{q}_1 + c_{212} \dot{q}_2 = 0 \\ c_{22} &= c_{221} \dot{q}_1 + c_{222} \dot{q}_2 = 0 \end{aligned}$$

**1.3** Consider matrix  $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{B}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ . Show, for the specific case of the given manipulator, that the following equality is true:  $\dot{\mathbf{q}}^T \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} = 0$

$$\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{B}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -m_2 l_2 s_2 \dot{q}_2 \\ -m_2 l_2 s_2 \dot{q}_2 & 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & -m_2 l_2 s_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_2 l_2 s_2 \dot{q}_2 \\ -m_2 l_2 s_2 \dot{q}_2 & 0 \end{bmatrix}$$

Therefore:

$$\dot{\mathbf{q}}^T \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} 0 & m_2 l_2 s_2 \dot{q}_2 \\ -m_2 l_2 s_2 \dot{q}_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = 0$$

<sup>2</sup> The general expression of the Christoffel symbols is:  $c_{ijk} = \frac{1}{2} \left( \frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i} \right)$

1.4 Prove that the equality  $\dot{q}^T N(q, \dot{q}) \dot{q} = 0$  is true for whatever manipulator.

Given the equation of the dynamic model of the manipulator:

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

we consider an energy conservation principle:

$$\frac{1}{2} \frac{d}{dt} (\dot{q}^T B(q) \dot{q}) = \dot{q}^T (\tau - g(q))$$

Taking the derivative at the left hand side and using the equation of the model:

$$\frac{1}{2} \frac{d}{dt} (\dot{q}^T B(q) \dot{q}) = \frac{1}{2} \dot{q}^T \dot{B}(q) \dot{q} + \dot{q}^T B(q) \ddot{q} = \frac{1}{2} \dot{q}^T (\dot{B}(q) - 2C(q, \dot{q})) \dot{q} + \dot{q}^T (\tau - g(q))$$

Combining the two equations the thesis follows.

## Exercise 2

Consider the problem of generating a trajectory for a scalar variable  $q(t)$  that interpolates a certain number of points at given instants.

**2.1** Explain the reasons why it is not advisable to use a single high order polynomial to interpolate such points.

The interpolation of  $n$  points with only one polynomial of degree  $n-1$  has several disadvantages:

1. finding the coefficients of the polynomial is computationally intensive and leads to ill-conditioned numerical problems, for large values of  $n$ ,
2. changing a single point  $(t_i, q_i)$  implies to compute again the entire polynomial;
3. adding a final point  $(t_{n+1}, q_{n+1})$  implies the use of a higher degree polynomial and thus the computation of all the coefficients;
4. the obtained solution in general presents undesired oscillations

**2.2** Consider now the interpolation of the following points:

|          |          |
|----------|----------|
| $t_1=0$  | $q_1=0$  |
| $t_2=3$  | $q_2=10$ |
| $t_3=6$  | $q_3=30$ |
| $t_4=8$  | $q_4=20$ |
| $t_5=10$ | $q_5=5$  |

Compute the values of the speed at the intermediate points, with an approximate method that makes use of the slopes in the intervals before and after each point.

We shall use the formula:

$$\dot{q}_k = \begin{cases} 0 & \text{sign}(R_k) \neq \text{sign}(R_{k+1}) \\ \frac{R_k + R_{k+1}}{2} & \text{sign}(R_k) = \text{sign}(R_{k+1}) \end{cases}$$

where:

$$R_k = \frac{q_k - q_{k-1}}{t_k - t_{k-1}}$$

is the slope in the interval  $[t_{k-1}, t_k]$ .

Then:

$$\dot{q}_2 = \frac{\frac{10}{3} + \frac{20}{3}}{2} = 5$$

$$\dot{q}_3 = 0$$

$$\dot{q}_4 = \frac{-\frac{10}{2} - \frac{15}{2}}{2} = -6.25$$

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### 2.3 Compute the expression of the cubic polynomial for the first interval (from $t_1$ to $t_2$ ).

We need to find the coefficients of the polynomial:

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

with the boundary conditions:

$$q(0) = 0 \quad q(3) = 10$$

$$\dot{q}(0) = 0 \quad \dot{q}(3) = 5$$

From the conditions at  $t = 0$ , we easily obtain  $a_0 = a_1 = 0$ . From the conditions at  $t = 3$  we obtain the linear system:

$$\begin{cases} 9a_2 + 27a_3 = 10 \\ 6a_2 + 27a_3 = 5 \end{cases} \Rightarrow \begin{cases} a_2 = 5/3 \\ a_3 = -5/27 \end{cases}$$

Then the expression of the cubic polynomial is:

$$q(t) = \frac{5}{3}t^2 - \frac{5}{27}t^3$$

### 2.4 Assume now that you want to interpolate the points given in item 2.2 of this exercise with the method of the splines. Explain how you can find the 16 conditions that allow you to determine the 4 cubic polynomials (one for each of the intervals).

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Since we have  $n = 5$  points, the conditions that can be imposed are:

- $2(n-1) = 8$  conditions of passage through the points
- $n-2 = 3$  conditions on continuity of velocities in the intermediate points
- $n-2 = 3$  conditions on continuity of accelerations in the intermediate points
- 2 conditions on the initial and final speeds

which sum to 16 conditions.

### Exercise 3

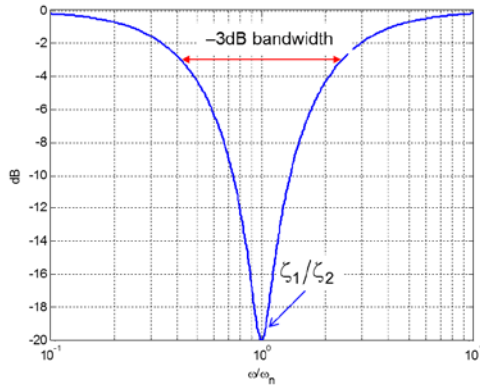
#### 3.1 Write the expression of the transfer function of a notch filter and sketch the amplitude Bode diagram of its frequency response.

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The transfer function of a notch filter takes this expression:

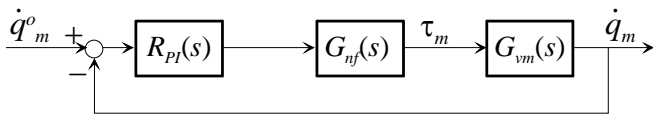
$$G_{nf}(s) = \frac{s^2 + 2\zeta_1\omega_n s + \omega_n^2}{s^2 + 2\zeta_2\omega_n s + \omega_n^2}$$

while the Bode diagram can be sketched as follows:



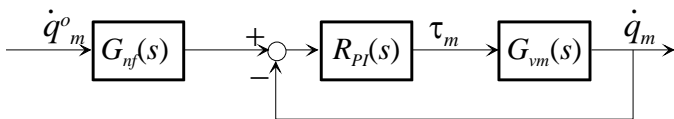
**3.2** Explain how a notch filter could be used in the control of a servomechanism.

In a motion control system the notch filter is usually inserted within the speed control loop, cascaded to the PI controller:

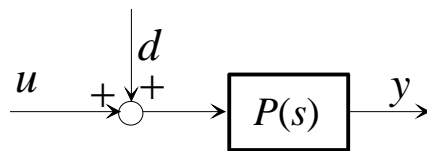


Often the filter is used to cancel a high frequency vibration, beyond the control bandwidth, which results into an acoustic noise (a whistle).

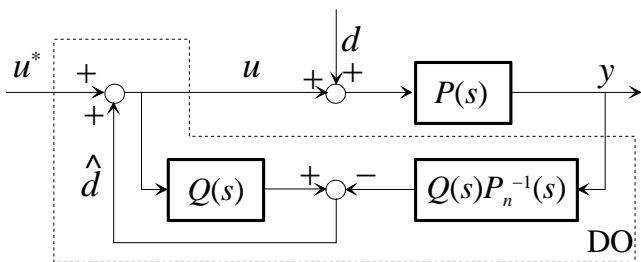
An alternative use of the notch filter is outside the speed control loop:



**3.3** Consider now a system affected by a load disturbance:



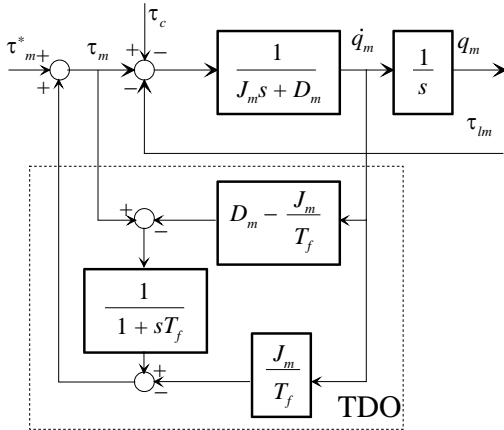
Sketch the block diagram of a disturbance observer.



**3.4** Assume now that the system under control is a motor, with transfer function from motor torque to speed given by:

$$P(s) = \frac{1}{J_m s + D_m}$$

Sketch the block diagram of a (torque) disturbance observer adapted for this case and explain what is the result that can be obtained with the application of such method.



Elaborating on the block diagram we have:

$$Q_m(s) = \frac{1}{J_m s^2 + D_m s} [\tau_m^*(s) - G_f(s)(\tau_c(s) + \tau_{lm}(s))]$$

with:

$$G_f(s) = \frac{sT_f}{1 + sT_f}$$

high pass filter. The torque disturbance is thus filtered quite effectively, especially in the low frequency range.

#### Exercise 4

Consider a manipulator where a system of forces is applied at the end-effector.

**4.1** Write the general expression of the dynamic model of the manipulator which includes the effect of such forces, explaining the meaning of each symbol used.

The expression of the dynamic model is:

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} - \mathbf{J}^T(\mathbf{q})\mathbf{h}$$

where:

$$\mathbf{h} = \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\mu} \end{bmatrix}$$

is the vector of the contact forces and moments applied by the end effector on the environment and  $\mathbf{J}$  is the geometrical Jacobian.

**4.2** Write the expression of an impedance controller for the whole manipulator and explain what is the result achieved with such controller.

The impedance controller is:

$$\boldsymbol{\tau} = \mathbf{B}(\mathbf{q})\mathbf{y} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) + \mathbf{J}^T(\mathbf{q})\mathbf{h}$$



with:

$$\mathbf{y} = \mathbf{J}_A^{-1}(\mathbf{q}) \mathbf{M}_d^{-1} (\mathbf{M}_d \ddot{\mathbf{x}}_d + \mathbf{D}_d \dot{\tilde{\mathbf{x}}} + \mathbf{K}_d \tilde{\mathbf{x}} - \mathbf{M}_d \dot{\mathbf{J}}_A(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} - \mathbf{h}_A)$$

where  $\tilde{\mathbf{x}} = \mathbf{x}_d - \mathbf{x}$ ,  $\mathbf{J}_A$  is the analytical Jacobian,  $\mathbf{h}_A$  is such that  $\mathbf{J}_A^T \mathbf{h}_A = \mathbf{J}^T \mathbf{h}$ ,  $\mathbf{M}_d$ ,  $\mathbf{D}_d$ ,  $\mathbf{K}_d$  are diagonal, positive definite matrices.

We obtain:

$$\mathbf{M}_d \ddot{\tilde{\mathbf{x}}} + \mathbf{D}_d \dot{\tilde{\mathbf{x}}} + \mathbf{K}_d \tilde{\mathbf{x}} = \mathbf{h}_A$$

i.e. a completely decoupled impedance relation.

#### 4.3 Write the expression of a rotational impedance.

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$$\mathbf{M}_\phi \Delta \ddot{\phi} + \mathbf{D}_\phi \Delta \dot{\phi} + \mathbf{K}_\phi \Delta \phi = \mathbf{T}^T(\phi_e) \boldsymbol{\mu}$$

where

$$\Delta \phi = \phi_d - \phi_e$$

is the error between a desired and an actual set of Euler angles.

#### 4.4 In the expression of a rotational impedance the desired values of a set of Euler angles appear. Briefly describe an alternative way to define the desired orientation and explain what are the advantages related to its adoption.

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The desired orientation can also be assigned with the axis/angle representation: if we assign two frames with the same origin but different orientations, it is always possible to determine a unit vector  $\mathbf{r}$  such that the second frame is obtained from the first one through a rotation of an angle  $\vartheta_r$  around the axis of such unit vector.

Basically the advantage is that the axis/angle representation is more intuitive.