POLITECNICO
MILANO 1863

Motion planning for mechanical systems

Academic year 2025-2026

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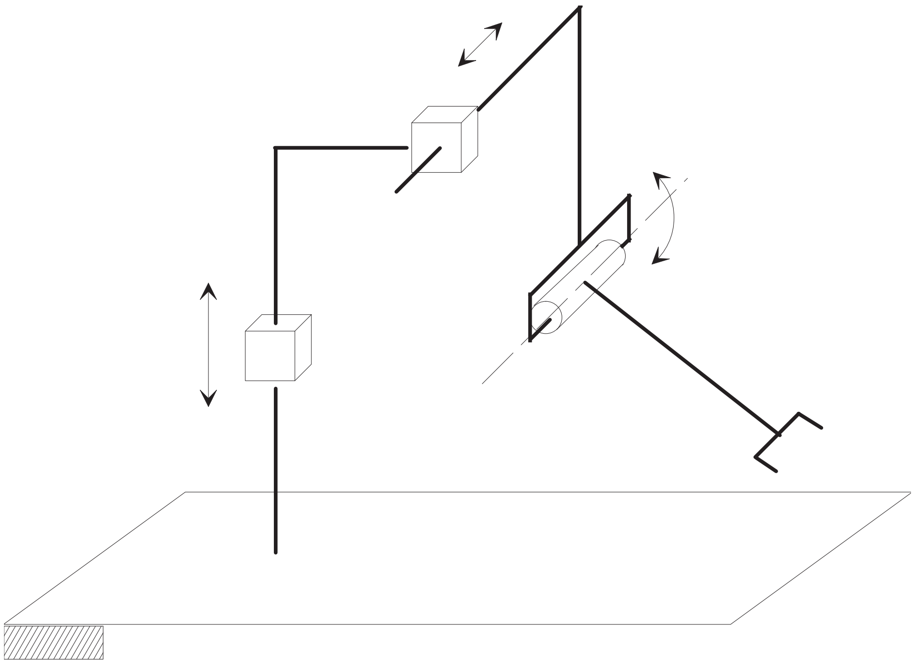
SOLUTIONS

Exercise 1

Consider the manipulator drawn in the figure:



Point 1.1 Plot, on the figure itself, the frames according to the Denavit-Hartenberg convention and fill in the relative table of parameters:

	a	α	d	ϑ
1				
2				
3				

An admissible choice of Denavit-Hartenberg frames is shown in the figure.

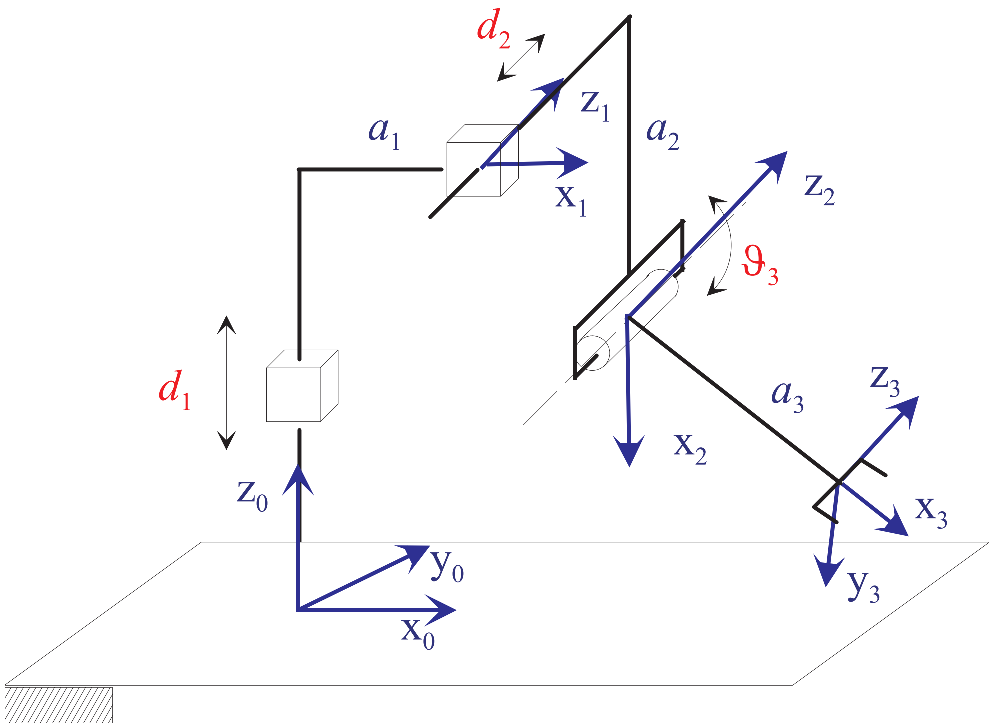
The Denavit-Hartenberg parameter table is as follows:

	a	α	d	ϑ
1	a_1	$-\frac{\pi}{2}$	d_1	0
2	a_2	0	d_2	$\frac{\pi}{2}$
3	a_3	0	0	ϑ_3

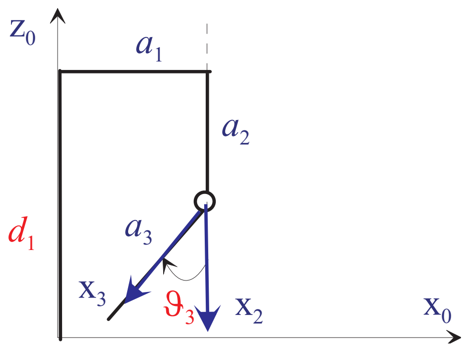
Point 1.2 For the given manipulator, write the equations of the direct kinematics relative to position only. [1]

[1] Remember, if you consider it useful for the solution of the exercise, the expression of the homogeneous transformation matrix between two consecutive triplets:

$$\mathbf{A}_i^{i-1} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} c_{\alpha_i} & s_{\vartheta_i} s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i} c_{\alpha_i} & -c_{\vartheta_i} s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



To calculate the direct kinematics, it is possible to obtain the 3 partial matrices of homogeneous transformation between consecutive frames and multiply them together, or proceed by inspection. Following the latter path, we can refer to the following representation in the $x_0 - z_0$ plane:



With simple trigonometric considerations we find that:

$$\begin{aligned} p_x &= a_1 - a_3 s_3 \\ p_z &= d_1 - a_2 - a_3 c_3 \end{aligned}$$

From the initial design of the robot, moreover, it can be recognized that:

$$p_y = d_2$$

Point 1.3 For the given manipulator, determine the geometric Jacobian (relative to linear velocities only), highlighting the singularity points[2]

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[2]Remember that the vector product between the vectors $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is $c = a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$

The Jacobian of position can be written as:

$$\mathbf{J}_P = \begin{bmatrix} \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 \times (\mathbf{p} - \mathbf{p}_2) \end{bmatrix}$$

For the calculation of this Jacobian, observe that:

$$\mathbf{z}_1 = \mathbf{z}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{p} - \mathbf{p}_2 = \begin{bmatrix} -a_3 s_3 \\ 0 \\ -a_3 c_3 \end{bmatrix}$$

Therefore, by performing the calculation of the vector product:

$$\mathbf{J}_P = \begin{bmatrix} 0 & 0 & -a_3 c_3 \\ 0 & 1 & 0 \\ 1 & 0 & a_3 s_3 \end{bmatrix}$$

The singularities are found by canceling the determinant of the Jacobian:

$$\det(\mathbf{J}_P) = a_3 c_3$$

Therefore the manipulator is in singular configuration when $\vartheta_3 = \frac{\pi}{2}$ *or* $\vartheta_3 = -\frac{\pi}{2}$.

Point 1.4 For a generic manipulator, explain why kinematic singularities are problematic. How many singularities does a 6-dof anthropomorphic manipulator typically have?

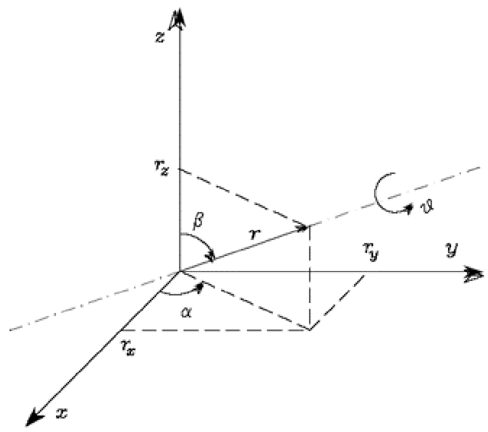
In a singularity the following things happen:

- *Loss of mobility (it is not possible to impose arbitrary motion laws)*
- *Possibility of infinite solutions to the kinematic inversion problem*
- *High velocities in joint space (around the singularity)*

An anthropomorphic manipulator typically has 3 singularities (shoulder, elbow, wrist).

Exercise 2

Point 2.1 Referring to the following figure, discuss the axis-angle representation of the orientation. In particular, explain how many parameters this representation is composed of and what relationship, if any, exists between these parameters.



Given two frames, the rotation of one relative to the other can be interpreted as a rotation of an angle ϑ around an axis $\mathbf{r}$. *The angle ϑ and the components of the vector* $\mathbf{r}$ *constitute the 4 parameters of the axis-angle representation. The link between these parameters is as follows:*

$$r_x^2 + r_y^2 + r_z^2 = 1$$

Point 2.2 Now suppose you want to plan the orientation trajectory of a robot's end-effector. The rotation matrices that express respectively the initial orientation and the final orientation with respect to the world frame are:

$$\mathbf{R}_i = \mathbf{I}_3, \qquad \mathbf{R}_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Determine the axis around which the rotation is to be made and the angle by which it is to be rotated.

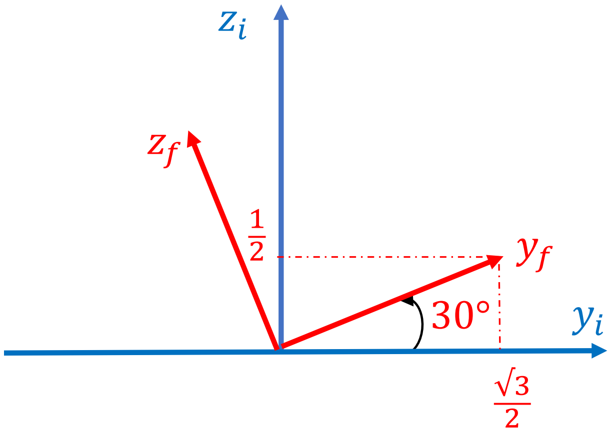
Clearly, since the initial orientation coincides with that of the world frame, the matrix $\mathbf{R}_f$ *already expresses the total rotation to be performed. This consists of an elementary rotation around the* $\mathbf{x}$ *axis of an angle equal to* $30°$*, as shown in the figure:*

Point 2.3 With reference to the data of this exercise, plan the evolution of the angle with a cubic time law, with positioning time equal to $T = 1s$. Assume the initial and final velocities to be null and express the angle in degrees.[3]

The cubic time law for position and angular velocity can be written as:

$$\begin{aligned} \vartheta(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ \dot{\vartheta}(t) &= a_1 + 2a_2 t + 3a_3 t^2 \end{aligned}$$

[3]If you have not been able to solve point 2 of the exercise, solve this point by assuming the final value of the angle 40°.



The boundary conditions to be imposed are:

$$\begin{aligned}\vartheta(0) &= 0\\ \vartheta(1) &= 30\\ \dot{\vartheta}(0) &= 0\\ \dot{\vartheta}(1) &= 0\end{aligned}$$

From the conditions at $t = 0$, it is easy to deduce that $a_0 = a_1 = 0$. We then obtain the system of equations:

$$\begin{aligned}a_2 + a_3 &= 30\\ 2a_2 + 3a_3 &= 0\end{aligned}$$

which solved gives $a_2 = 90$, $a_3 = -60$. Therefore the time law to be imposed on the corner is:

$$\vartheta(t) = 90t^2 - 60t^3$$

Point 2.4 With the cubic profile previously used, are the angle, the related velocity, and the acceleration, continuous in all the time instants, including the initial and the final ones? Comment your answer.

While the position and the velocity are continuous in all points, including the distal points, the acceleration is not continuous at the beginning and at the end of the trajectory. In order to achieve continuity of the acceleration as well, a fifth-order polynomial needs to be used.

Exercise 3

Point 3.1 For a single rigid body, write the expression of the kinetic energy, specifying the meaning of all the symbols used.

The kinetic energy of a rigid body can be expressed with the König theorem as follows:

$$T = \frac{1}{2} m \dot{\mathbf{p}}_c^T \dot{\mathbf{p}}_c + \frac{1}{2} \omega \mathbf{I}_c \omega$$

where $\dot{\mathbf{p}}_c$ *is the velocity of the center of mass,* ω *is the angular velocity of the body,* m *is the mass, and* $\mathbf{I}_c$ *is the inertia tensor relative to the center of mass expressed in a frame parallel to the reference frame with origin at the center of mass.*

Point 3.2 For a generic manipulator, write the expression of the kinetic energy. What are the properties of the inertia matrix?

The kinetic energy of a whole manipulator is given by:

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{B}(\mathbf{q}) \dot{\mathbf{q}}$$

where $\mathbf{B}(\mathbf{q})$ *is the inertia matrix of the robot. Such matrix depends on* $\mathbf{q}$ *and is symmetric and positive definite.*

Point 3.3 In the derivation of the Euler-Lagrange model of a systems of bodies, the Lagrange's equations are used. Write the general expression of such equations, defining in particular the concept of Lagrangian.

The Lagrange's equations can be written as follows:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \xi_i, \qquad 1 = 1, \ldots, n$$

where q_i *is the* i_{th} *generalized coordinate,* ξ_i *is the* i_{th} *generalized external action, while* $L = T - U$ *is the Lagrangian of the system, difference between the kinetic and the potential energies.*

Point 3.4 Write the complete dynamic model of the robot, as derived from the Euler-Lagrange method. Specify the meaning of the symbols used.

The complete dynamic model can be written as:

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \tau$$

where $\mathbf{B}(\mathbf{q})$ *is the inertia matrix,* $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ *is the matrix of Coriolis and centrifugal terms,* $\mathbf{g}(\mathbf{q})$ *are the gravitational terms, and* τ *are the torques applied at the joints.*