POLITECNICO MILANO 1863

Motion planning for mechanical systems

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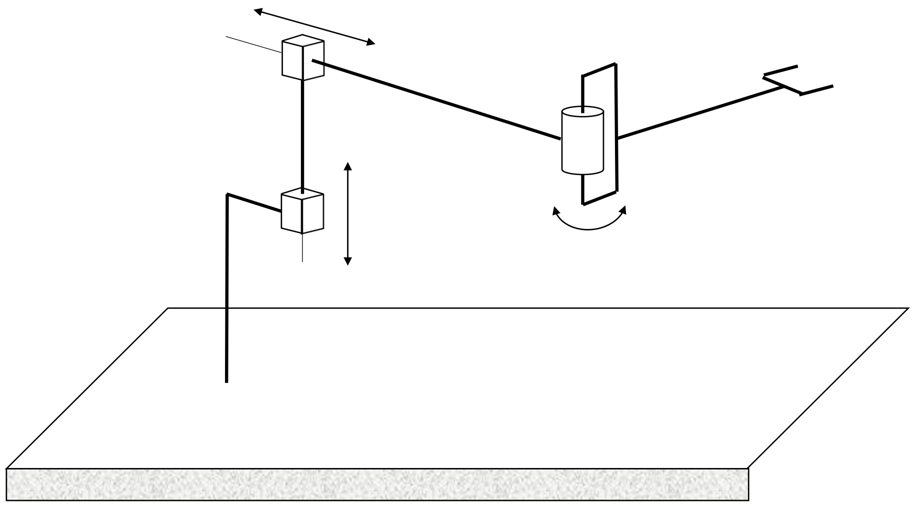
SOLUTIONS

Exercise 1

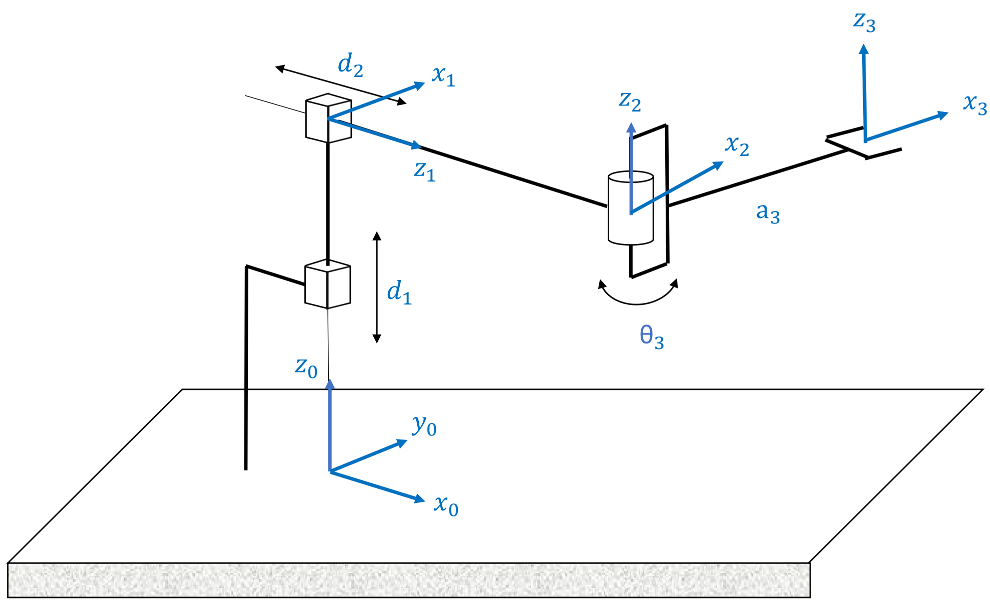
Consider the manipulator drawn in the figure:



Point 1.1 Plot, on the figure itself, the frames according to the Denavit-Hartenberg convention and fill in the relative table of parameters:

	a	α	d	ϑ
1				
2				
3				

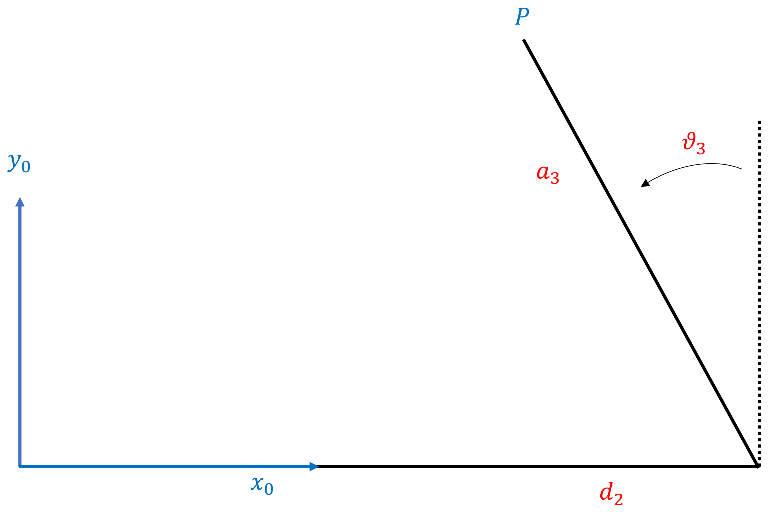
An admissible choice of Denavit-Hartenberg frames is shown in the figure.
The Denavit-Hartenberg parameter table is as follows:



	a	α	d	ϑ
1	0	90°	d_1	90°
2	0	−90°	d_2	0
3	a_3	0	0	ϑ_3

Point 1.2 For the given manipulator, write the equations of the direct kinematics relative to position only. [1]

To calculate the direct kinematics, it is possible to obtain the 3 partial matrices of homogeneous transformation between consecutive frames and multiply them together, or proceed by inspection. Following the latter path, we can refer to the following representation in the plane $x_0 - y_0$:



With simple trigonometric considerations we find that:

$$\begin{aligned} p_x &= d_2 - a_3 s_3 \\ p_y &= a_3 c_3 \end{aligned}$$

From the initial design of the robot, it can be recognized that:

$$p_z = d_1$$

Point 1.3 For the given manipulator, determine the geometric Jacobian (relative only to linear velocities), highlighting the singularity points

The Jacobian of position can be written as:

$$\mathbf{J}_P = \left[\begin{array}{ccc} \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 \times (\mathbf{p} - \mathbf{p}_2) \end{array}\right]$$

[1] Remember, if you consider it useful for the solution of the exercise, the expression of the homogeneous transformation matrix between two consecutive frames is:

$$\mathbf{A}_i^{i-1} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} c_{\alpha_i} & s_{\vartheta_i} s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i} c_{\alpha_i} & -c_{\vartheta_i} s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the calculation of this Jacobian, observe that:

$$\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{p} - \mathbf{p}_2 = \begin{bmatrix} d_2 - a_3 s_3 - d_2 \\ a_3 c_3 \\ d_1 \end{bmatrix}$$

Therefore, by performing the calculation of the vector product:

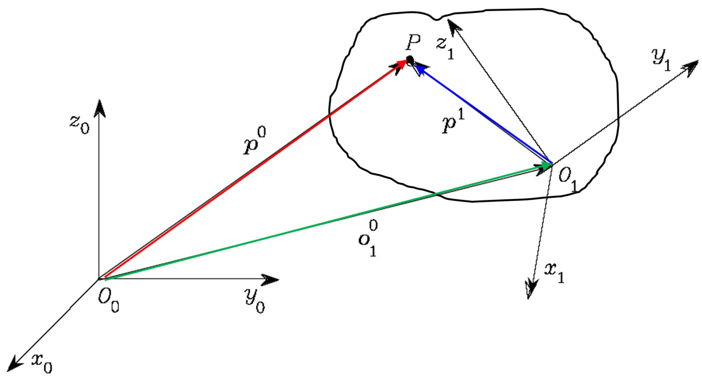
$$\mathbf{J}_P = \begin{bmatrix} 0 & 1 & -a_3 c_3 \\ 0 & 0 & -a_3 s_3 \\ 1 & 0 & 0 \end{bmatrix}$$

Singularities are found by setting the determinant of the Jacobian equal to 0:

$$\det(\mathbf{J}_P) = -a_3 s_3$$

Therefore the manipulator is in singular configuration when $s_3 = 0$, *i.e.* $\vartheta_3 = 0$ *or* $\vartheta_3 = \pi$.

Point 1.4 Making reference to the following picture, write the expression of vector $\mathbf{p}^0$ in terms of vector $\mathbf{p}^1$.



Vector $\mathbf{p}^0$ *can be written in terms of homogeneous coordinates as follows:*

$$\begin{bmatrix} \mathbf{p}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{o}_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}^1 \\ 1 \end{bmatrix}$$

where:

$$\begin{bmatrix} \mathbf{R}_1^0 & \mathbf{o}_1^0 \\ 0 & 1 \end{bmatrix}$$

is the homogeneous transformation matrix of frame 1 with respect to frame 0.

Exercise 2

Point 2.1 Consider a trapezoidal velocity profile trajectory. Derive the velocity expression $\dot{q}_v$ in the middle section (cruising velocity) given the positioning time T, the acceleration time T_a and the distance to travel h.

The distance traveled is the velocity integral, i.e. the area of the velocity trapezoid. Calculating this area we obtain:

$$h = \frac{(T + T - 2T_a)\dot{q}_v}{2} = (T - T_a)\dot{q}_v$$

from which:

$$\dot{q}_v = \frac{h}{T - T_a}$$

Point 2.2 Now consider generating the trajectory with trapezoidal velocity profile for a variable $q(t)$ with the following data:

$$\begin{array}{cc} q_i = 0 & q_f = 40^\circ \\ \dot{q}_{\max} = 10^\circ/s & \ddot{q}_{\max} = 10^\circ/s^2 \end{array}$$

Determine the minimum positioning time T and its acceleration time T_a

Since it results:

$$\frac{\dot{q}_{\max}^2}{\ddot{q}_{\max}} = \frac{100}{10} = 10 < h = 40$$

A trapezoidal velocity profile can be determined. The following is therefore obtained as a positioning time:

$$T = \frac{h}{\dot{q}_{\max}} + \frac{\dot{q}_{\max}}{\ddot{q}_{\max}} = \frac{40}{10} + \frac{10}{10} = 5$$

The acceleration time is instead valid:

$$T_a = \frac{\dot{q}_{\max}}{\ddot{q}_{\max}} = \frac{10}{10} = 1$$

Point 2.3 With reference to the planning in the previous point, suppose that the positioning time has increased by 20%, without changing the acceleration time. Determine the cruising velocity that is obtained as a result of the change in the positioning time.

The new positioning time will be $T_{new} = 1.2 \cdot T = 6$. *Using the formula derived in the first point of this exercise we obtain:*

$$\dot{q}_v = \frac{h}{T_{new} - T_a} = \frac{40}{6-1} = 8$$

which corresponds to 80% *of the maximum speed.*

Point 2.4 Consider now the planning of a position trajectory in the operational space. Explain how a time law like the one elaborated in the previous points can be exploited in such planning.

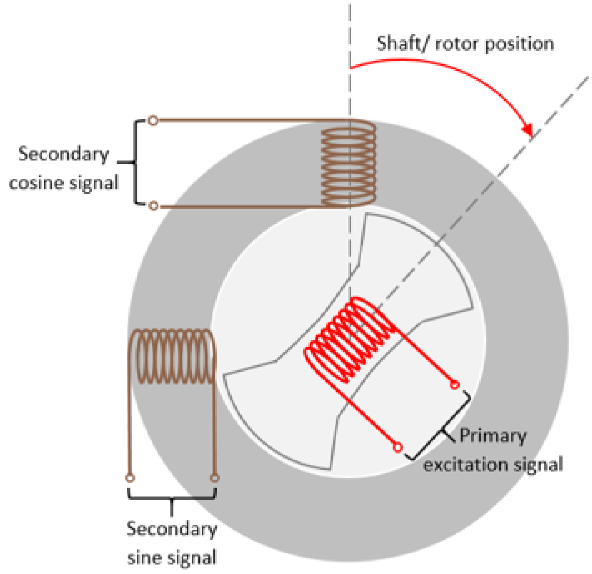
In planning the trajectory for the position of the end-effector, once a geometric path $\mathbf{p} = \mathbf{p}(s)$ *has been defined, the time law is assigned to the natural coordinate* s. *Therefore the trapezoidal velocity profile can be used as time law for* $s = s(t)$.

Exercise 3

Point 3.1 Explain why in robotics dedicated systems to measure the joint positions are used.

In order to control the motion of the robot in closed loop, feedback of the joint positions is needed.

Point 3.2 Referring to the following figure, briefly explain (without resorting to formulas) the operating principle of a resolver



The excitation winding is arranged on the resolver rotor, two spatial quadrature windings (pick-ups) are arranged on the stator. The rotor circuit is excited with a high-frequency alternating voltage, while the two stator circuits collect, by mutual induction, voltages that are proportional respectively to the sine and cosine of the angle of rotation. By processing these measurements, it is possible to obtain the information relating to the angle.

Point 3.3 Now consider an absolute encoder. On the basis of the operating principle, derive the formula that expresses the resolution of the instrument. Then determine the minimum number of tracks to have a resolution of less than a tenth of a degree.

The resolver consists of concentric crowns: the first subdivides the turn angle into two flat angles, the second each flat angle into two 90-degree angles, and so on. With N *tracks we will therefore have* 2^N *subdivisions of the round angle and therefore the resolution is equal to* $\frac{360°}{2^N}$.
Proceeding by trial and error, we obtain that for $N = 10$ *the resolution is* $\frac{360°}{1024} = 0.35°$, *for* $N = 10$ *the resolution is* $\frac{360°}{2048} = 0.17°$, *for* $N = 12$ *the resolution is* $\frac{360°}{4096} = 0.088°$. *Therefore, at least 12 tracks are needed to obtain a resolution of one tenth of a degree.*

Point 3.4 An indirect measurement of speeds can also be obtained from encoder measurements. Explain the fixed-time and fixed-space methods used for this purpose.

In the fixed-time method, the output of the time-step encoder fixed T_s is sampled:

$$\omega(k) = \frac{\theta(k) - \theta(k-1)}{T_s}$$

In the fixed-space method, the time required to change the encoder output by one bit ($\Delta\theta$ variation) is measured:

$$\omega(k) = \frac{\Delta\theta}{t_k - t_{k-1}}$$