Industrial automation and robotics

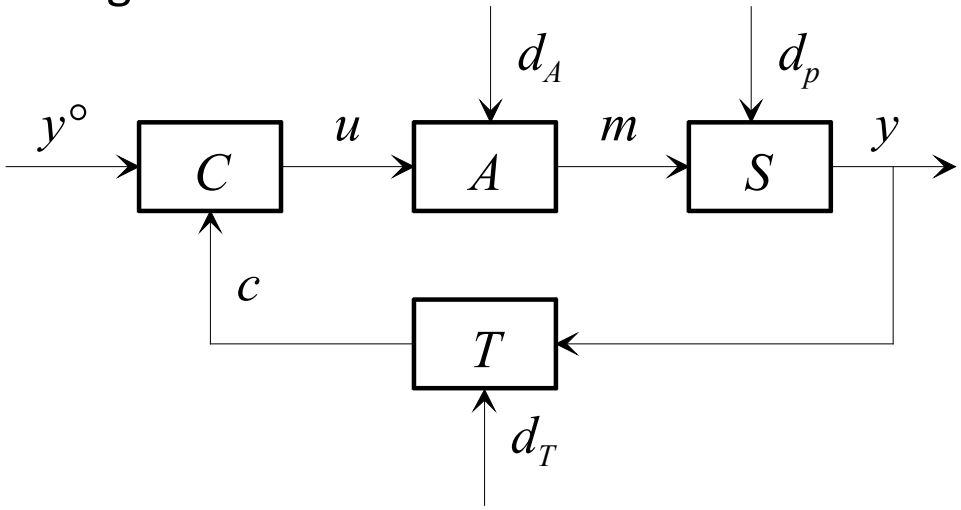
Feedback control systems

Prof. Paolo Rocco (paolo.rocco@polimi.it)

Politecnico di Milano, Dipartimento di Elettronica, Informazione e Bioingegneria

Elements of a control system

A generic **closed loop control system** can be represented by the following block diagram:

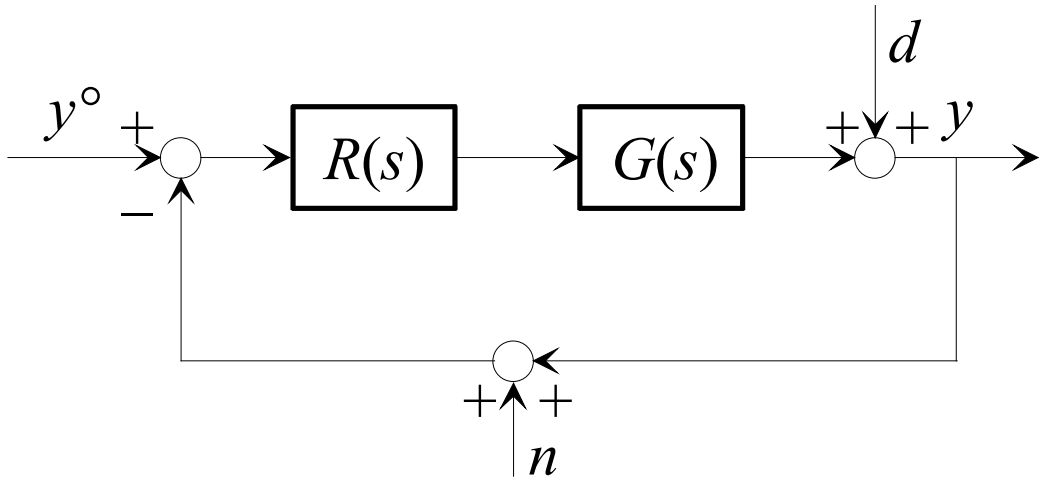


- S: system under control (or process)
- T: transducer
- A: actuator
- C: controller (or regulator)

- y: controlled variable
- y^o: set point
- c: measurement
- u: control variable
- m: actuation variable
- d_A: disturbance on the actuator
- d_P: disturbance on the process
- d_T: disturbance on the transducer

A linear feedback control system

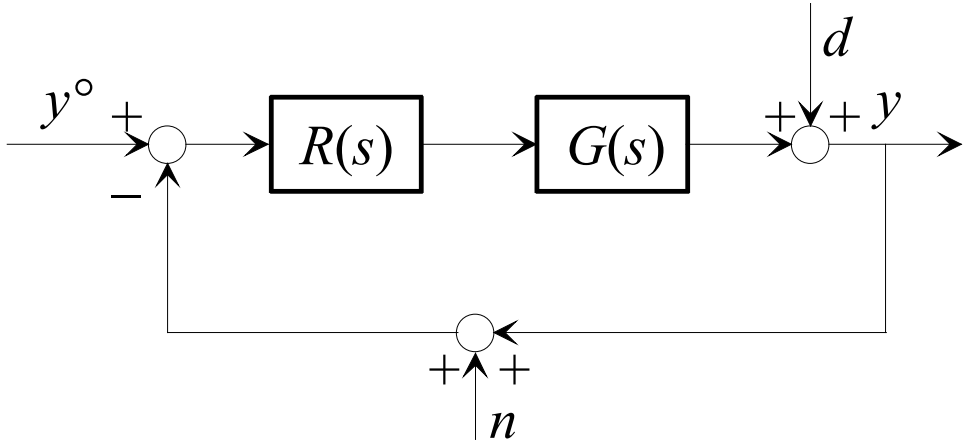
A **linear feedback control system** can be represented by the following block diagram:



where:

- $G(s)$ is the transfer function of the system under control (including effects of the transducer and the actuator)
- $R(s)$ is the transfer function of the controller

Design of the controller



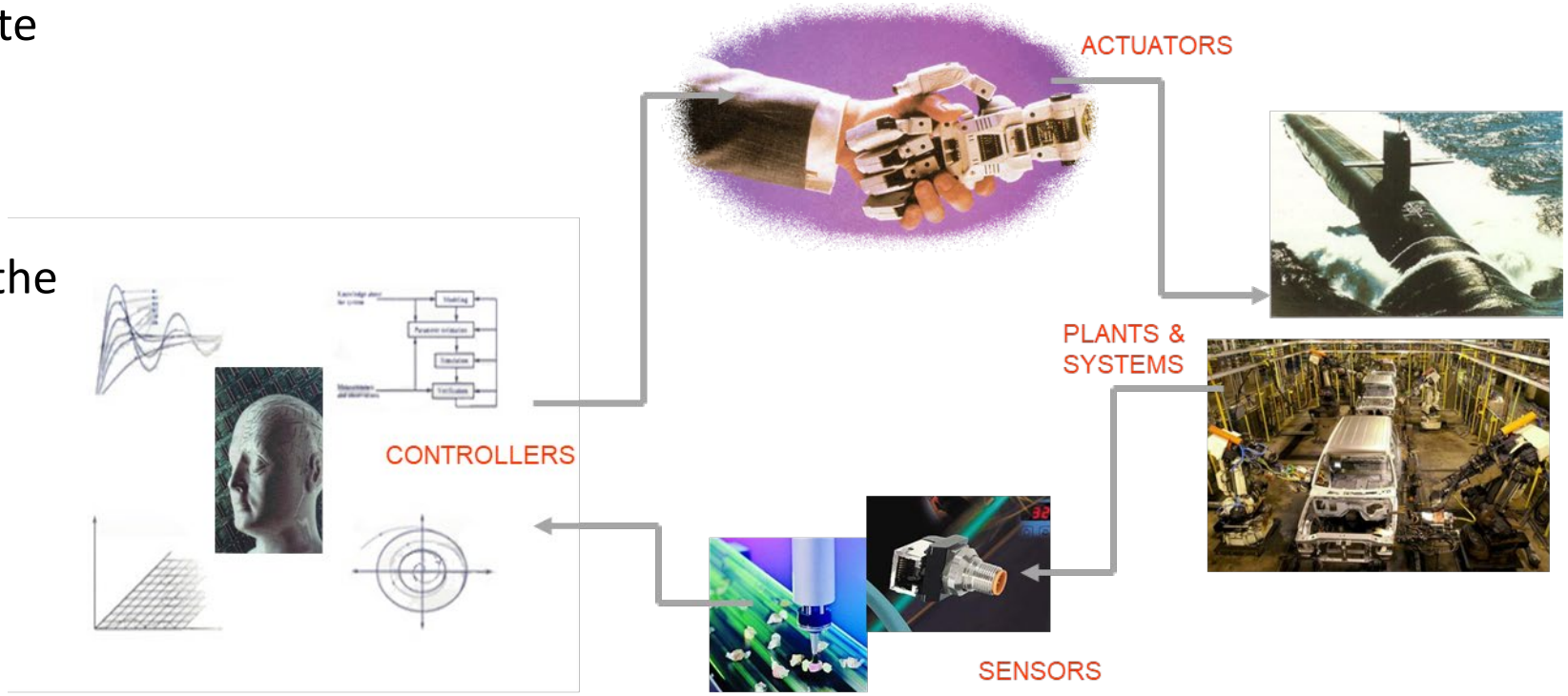
The design of the controller consists in determining the transfer function $R(s)$ of the controller, given the transfer function $G(s)$ of the system under control, so as to satisfy some **design specifications**:

- Stability
- Dynamic performance (how fast the control system is in tracking the reference value)
- Rejection of disturbances (the control system has to guarantee performance even against disturbances)
- Steady-state performance (at steady-state the error should be zero or anyway limited)

The role of feedback

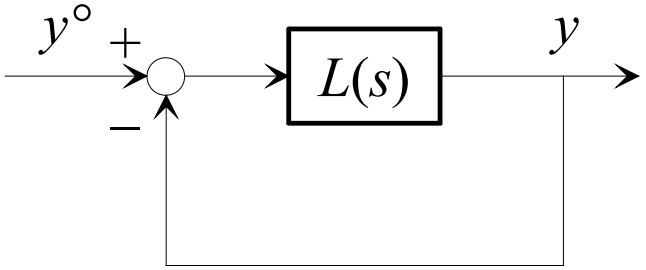
Feedback control means to take decisions based on measurements of some variables and some optimization criterion (steady-state and dynamic performance, disturbance rejection…)

We will see in the following how the different specifications can be addressed



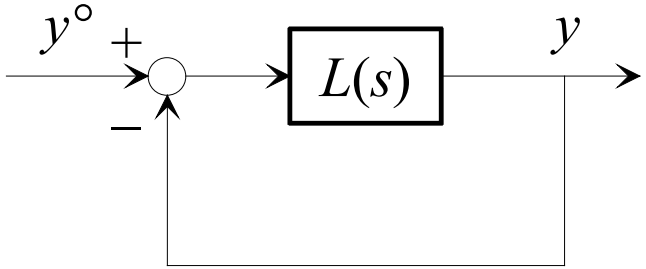
Stability of the control system

We highlight the loop transfer function $L(s) = R(s)G(s)$ and neglect the disturbances, which are not relevant for the discussion of stability:



The problem arises in terms of studying the **stability of the closed loop system**, knowing $L(s)$.

Characteristic polynomial



We express the loop transfer function as a ratio of polynomials: $L(s) = \dfrac{N(s)}{D(s)}$

The transfer function from y° to y takes the expression :

$$\frac{y}{y^\circ} = \frac{L(s)}{1 + L(s)} = \frac{\dfrac{N(s)}{D(s)}}{1 + \dfrac{N(s)}{D(s)}} = \frac{N(s)}{N(s) + D(s)}$$

We define the denominator of this transfer function **closed loop characteristic polynomial**: $\chi(s) = N(s) + D(s)$

Characteristic polynomial

$$\chi(s) = N(s) + D(s)$$

The closed loop system is **asymptotically stable** if and only if all the roots of the characteristic polynomial in closed loop have a negative real part

Example: $$L(s) = \frac{s^2 - s + 1}{s^3 + s^2 + s + 1} \quad \Rightarrow \quad \begin{aligned} \chi(s) &= s^2 - s + 1 + s^3 + s^2 + s + 1 \\ &= s^3 + 2s^2 + 2 \end{aligned}$$

It violates the necessary condition: the system is not asymptotically stable

- The characteristic polynomial criterion **does not lend itself to the synthesis of the regulator**, i.e. to determine $R(s)$ so that the closed loop system is asymptotically stable
- Other criteria, based on the frequency response, are better for this

Bode criterion

The **Bode criterion** is a graphical criterion for closed loop system stability that is based on Bode plots of the frequency response associated with the loop transfer function $L(s)$.

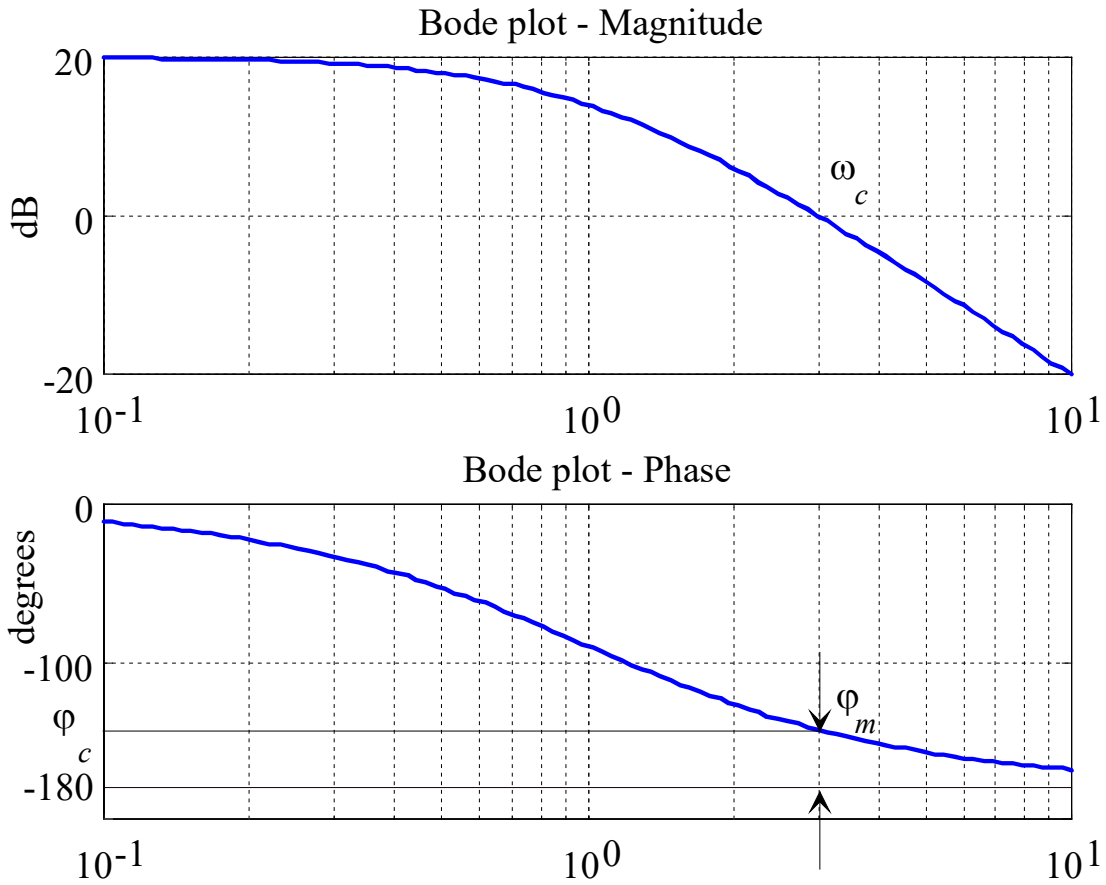
It is valid if some **conditions of applicability** are met:

- $L(s)$ has positive dc gain
- $L(s)$ has no positive real-part poles
- The Bode diagram of the magnitude of $L(j\omega)$ intersects the axis at 0 dB once and only once

Bode criterion

In order to apply the criterion, a number of **definitions** must be introduced:

- Crossover frequency ω_c : ω: $|L(j\omega)| = 1$
- Critical phase φ_c : $\varphi_c = \angle L(j\omega_c)$
- Phase margin φ_m : $\varphi_m = 180° - |\varphi_c|$



Bode criterion: the closed loop system is asymptotically stable **if and only if the phase margin is positive**

$$\boxed{\varphi_m > 0}$$

Bode criterion

Example:

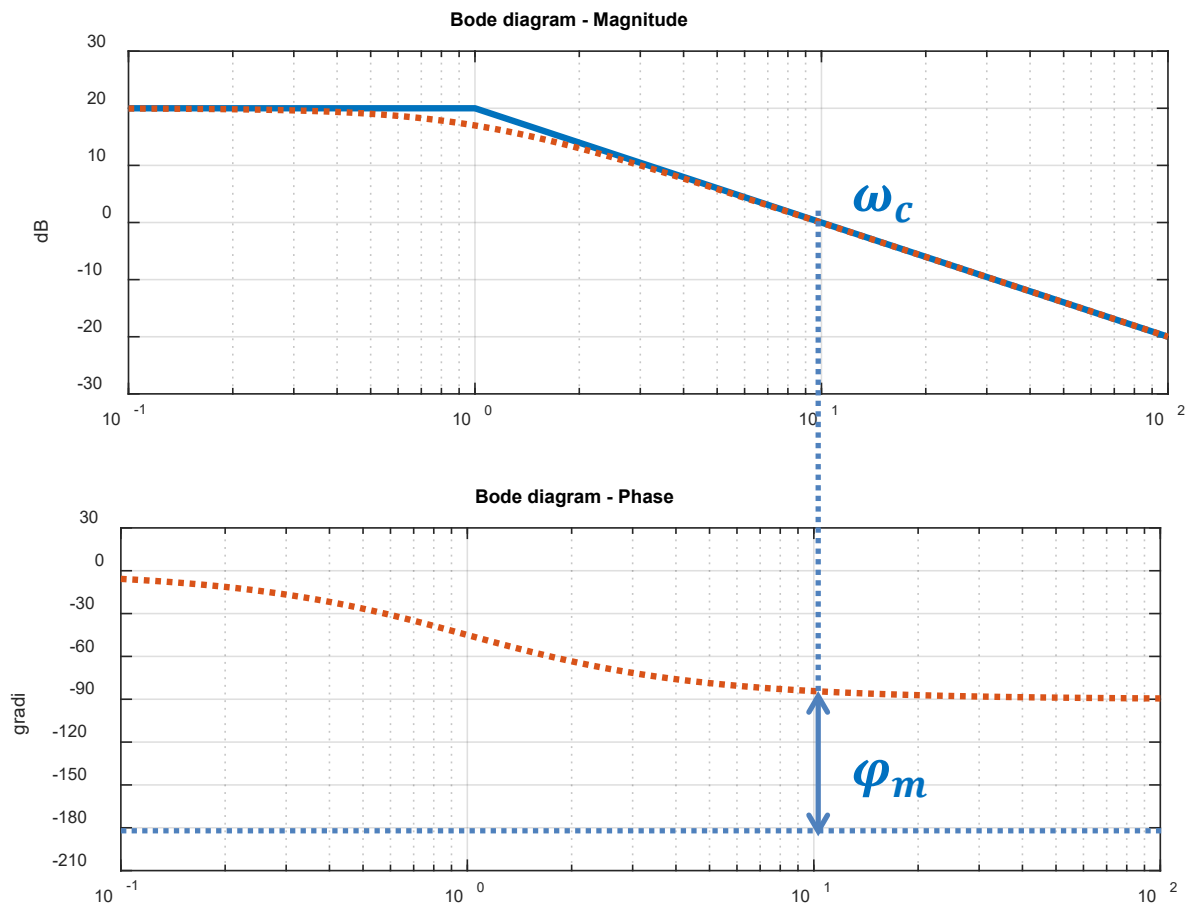
$$L(s) = \frac{10}{1+s}$$

The crossover frequency ω_c is $\cong$ 10 rad/s

The phase margin φ_m is $\cong 90°$

The closed loop system is then **asymptotically stable**

In general, if the slope around ω_c is -20 db/decade, the closed loop system has a good phase margin



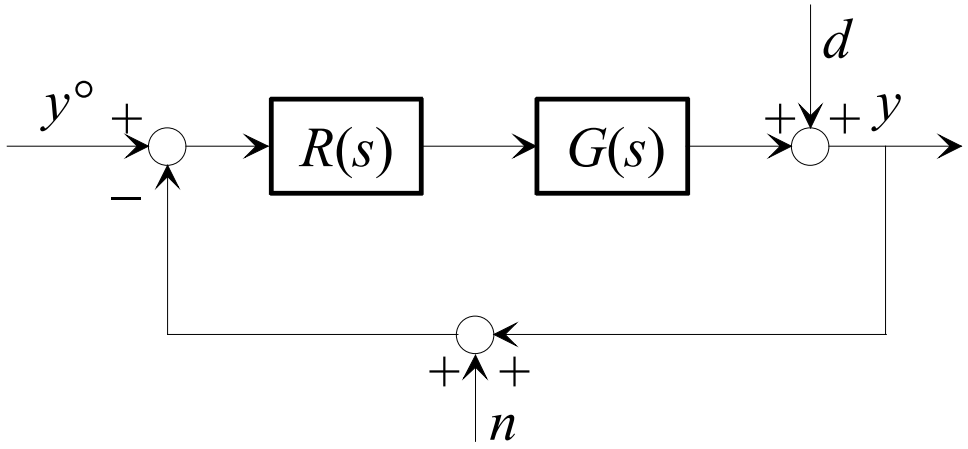
Dynamic performance

Dynamic performance refers to the behavior of the control system during transients.
In particular, we are interested in:

- **Response rate**: the speed with which the controlled variable follows abrupt variations (e.g. step) of the reference
- **Rejection of disturbances**: ability of the control system to track the reference even in the presence of disturbances

Dynamic performance

Consider the block diagram of the control system:



$$\frac{Y(s)}{Y^\circ(s)} = F(s) = \frac{R(s)G(s)}{1 + R(s)G(s)}$$ complementary sensitivity function

$$\frac{Y(s)}{D(s)} = S(s) = \frac{1}{1 + R(s)G(s)}$$ sensitivity function

Dynamic performance

Assume for a moment that the controller is just a gain K:

$$\frac{Y(s)}{Y^{\circ}(s)} = F(s) = \frac{KG(s)}{1+KG(s)} \rightarrow 1, \qquad K \rightarrow \infty$$

$$\frac{Y(s)}{D(s)} = S(s) = \frac{1}{1+KG(s)} \rightarrow 0, \qquad K \rightarrow \infty$$

It seems convenient to take a gain K as large as possible, to obtain perfect tracking of the reference and rejection of the disturbance

However:

- There are stability issues
- The controller is a complete transfer function, not just a gain, and we need to better understand how to proceed

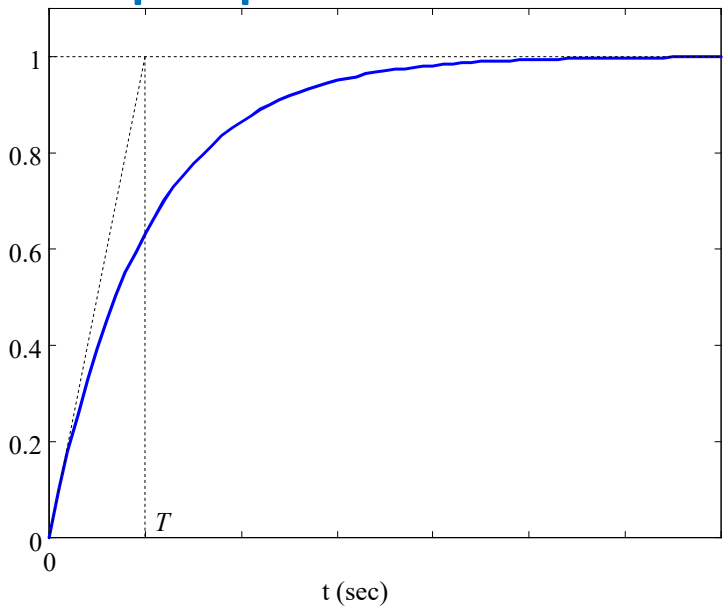
- The rejection of the measurement disturbance is also relevant: $\frac{Y(s)}{N(s)} = -F(s)$

Response speed in the frequency domain

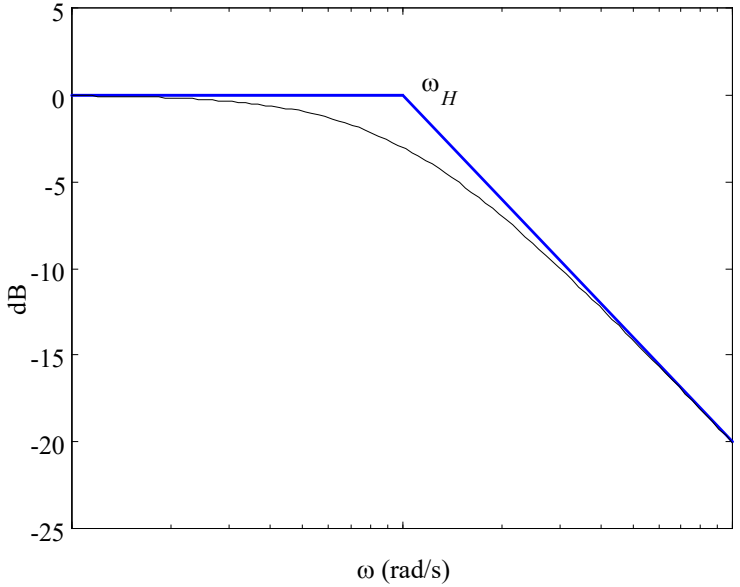
The concept of the response speed of a dynamical system can also be expressed in the frequency domain.
Let's consider a time-constant system:

$$H(s) = \frac{1}{1 + sT}, \qquad T > 0$$

Step response



Bode plot



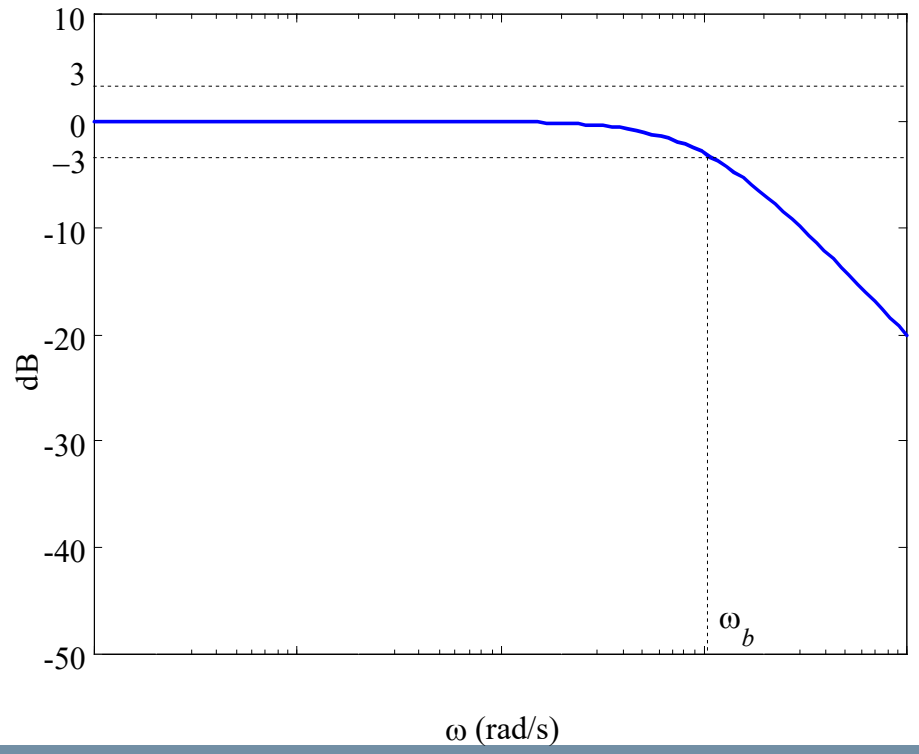
The faster the system, the smaller the time constant T, i.e. the wider the bandwidth defined by $\omega_H = 1/T$

Bandwidth of the control system

Consider again the transfer function from reference y^o to controlled variable Y :

$$\frac{Y(s)}{Y^o(s)} = F(s)$$

A reasonable Bode plot of the magnitude of F:



The **upper end of the bandwidth** ω_b is an **indicator of the response speed**: the larger ω_b, the more reactive the system is.

Bandwidth and crossover frequency

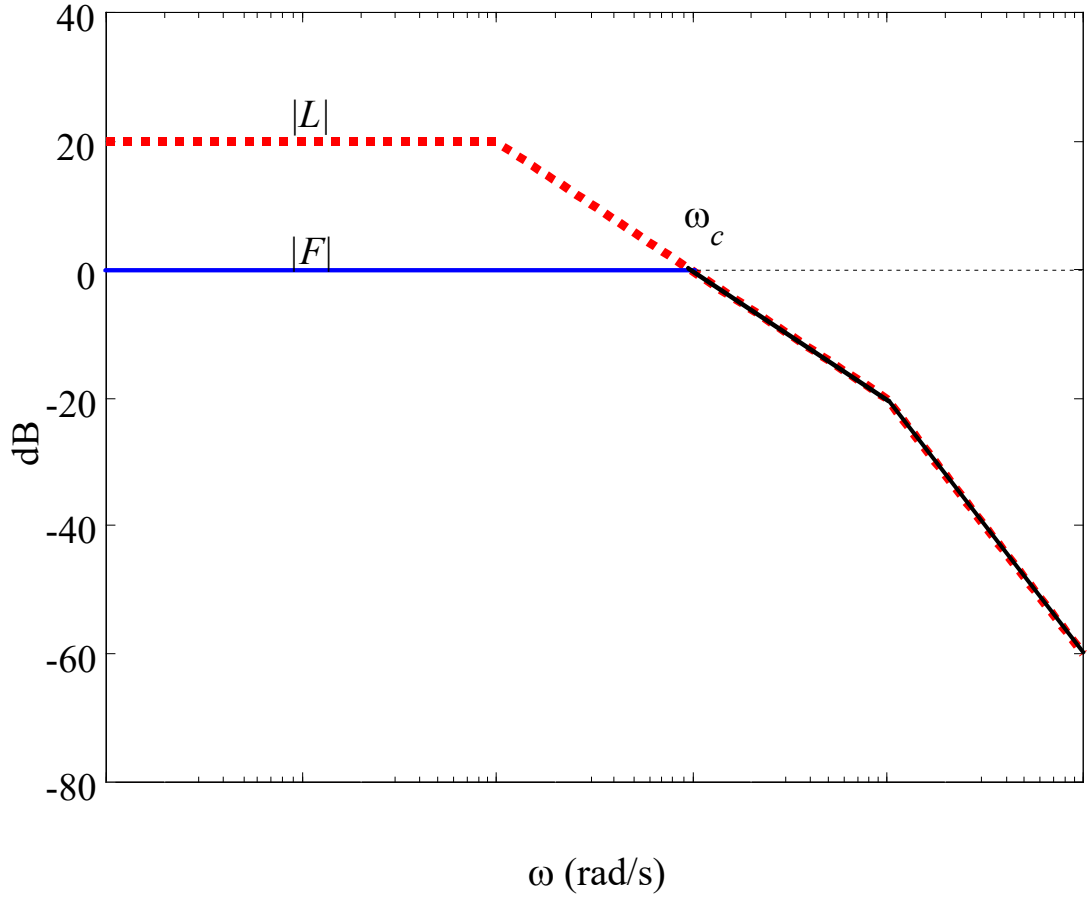
Consider the relation between the closed loop transfer function $F(s)$ and the loop transfer function $L(s)$:

$$|F(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|} \approx \begin{cases} 1 & \forall\omega : |L(j\omega)| >> 1 \\ |L(j\omega)| & \forall\omega : |L(j\omega)| << 1 \end{cases}$$

or:

$$|F(j\omega)| \approx \begin{cases} 1 & \forall\omega << \omega_c \\ |L(j\omega)| & \forall\omega >> \omega_c \end{cases}$$

This graphical approximation (valid for large values of the phase margin) suggests that a good indicator of the extension of the bandwidth of the closed-loop system is the **crossover frequency** ω_c

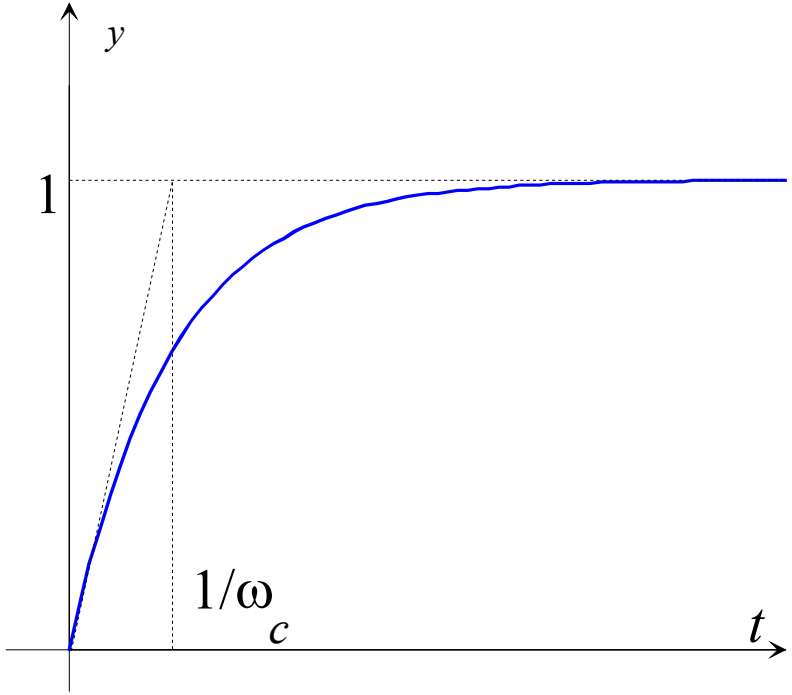


Bandwidth and crossover frequency

In a first approximation, the closed loop transfer function can be expressed as a **first order system** with unitary gain and a pole at frequency ω_c

$$F(s) \approx \frac{1}{1 + \frac{s}{\omega_c}}$$

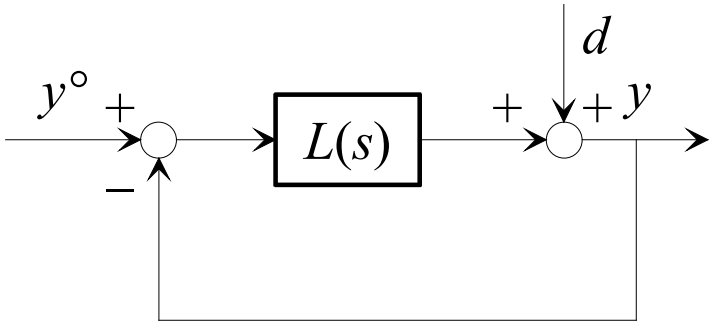
The step response is then:



and practically reaches the steady-state value in a time $\approx 4 \div 5 \frac{1}{\omega_c}$

Rejection of disturbances

Let's consider a disturbance in the forward path:



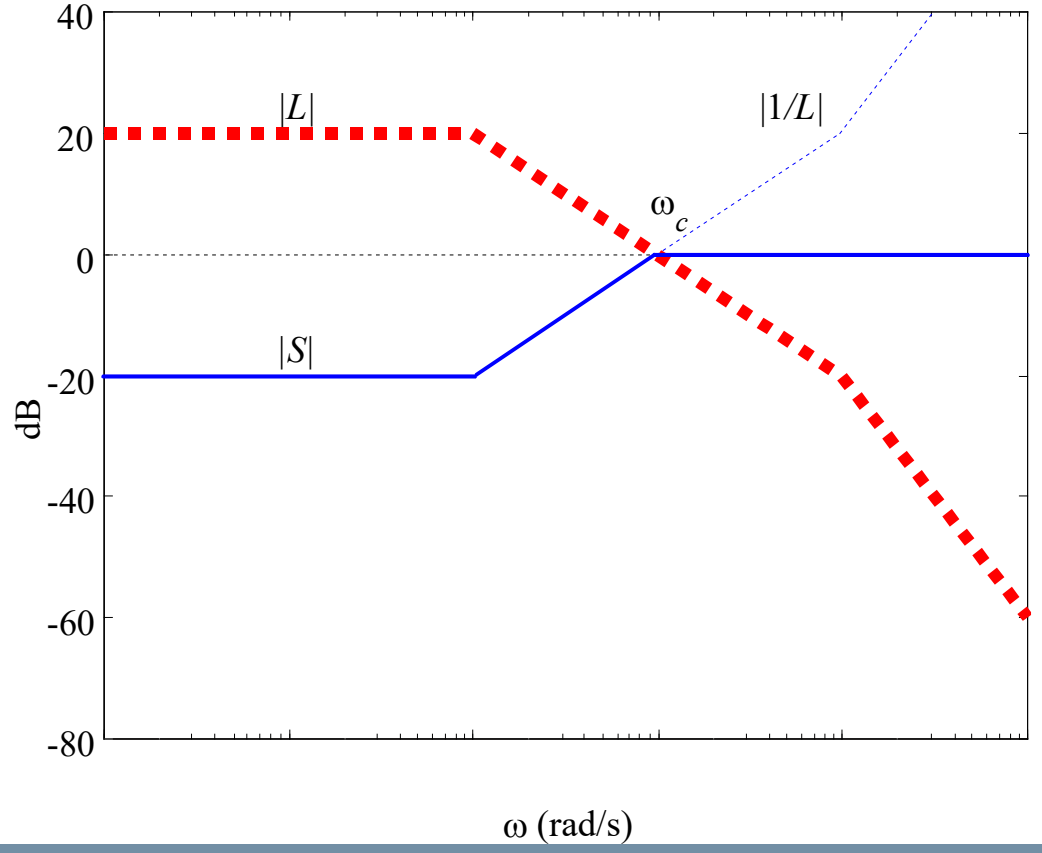
$$\frac{Y(s)}{D(s)} = S(s) = \frac{1}{1 + L(s)}$$

This transfer function is called **sensitivity function**

Rejection of disturbances

An approximation:

$$|S(j\omega)| = \frac{1}{|1+L(j\omega)|} \approx \begin{cases} \dfrac{1}{|L(j\omega)|} & \forall\omega << \omega_c \\ 1 & \forall\omega >> \omega_c \end{cases}$$



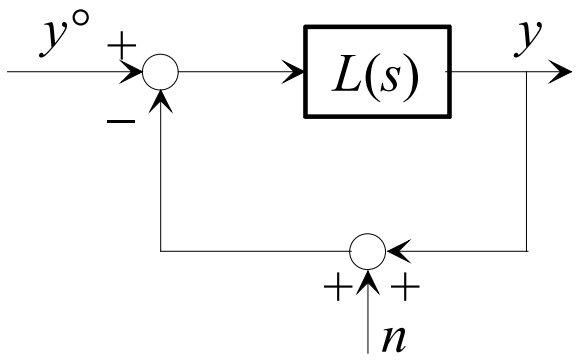
The harmonic components of the disturbance inside the bandwidth are attenuated on the controlled variable.

Therefore:

- the **bandwidth must be wide enough** to hold the significant harmonics of the disturbance
- the higher the magnitude of L in bandwidth, the greater the attenuation

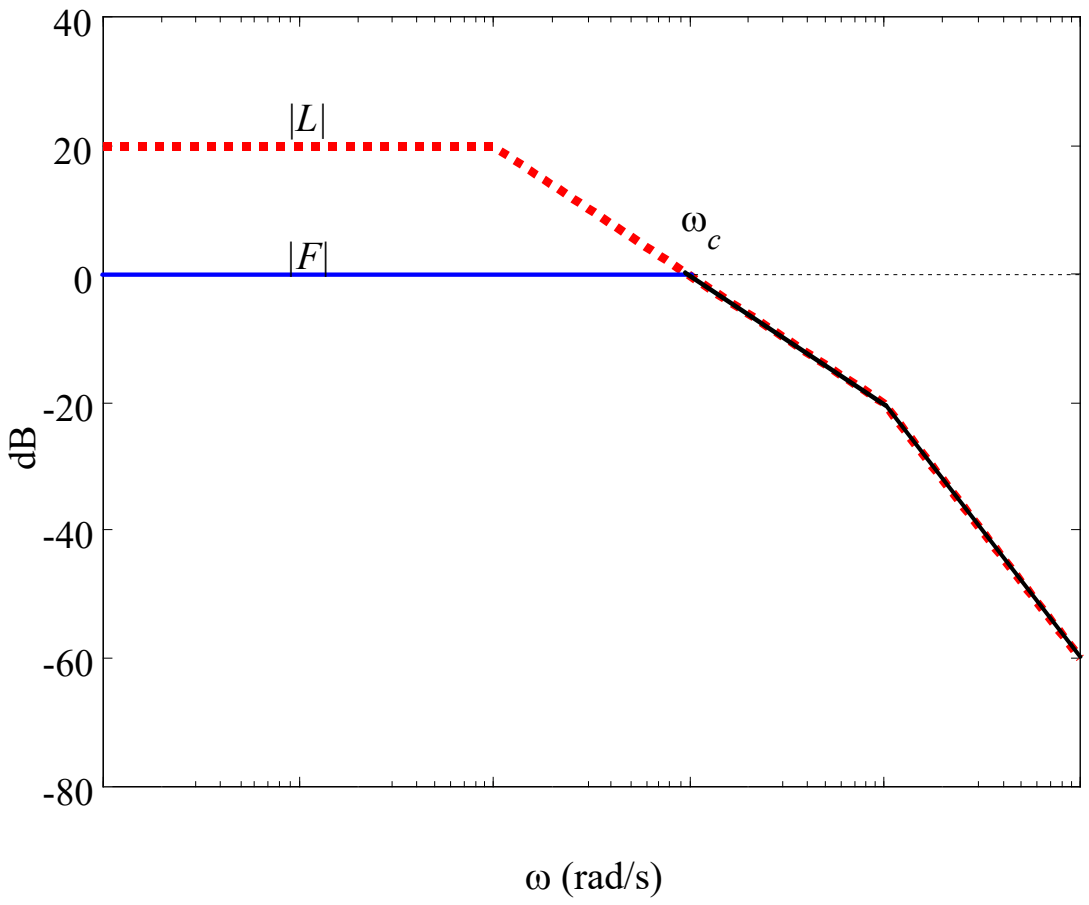
Rejection of disturbances

Consider now a disturbance in the feedback path:



$$\frac{Y(s)}{N(s)} = -F(s) = -\frac{L(s)}{1 + L(s)}$$

Rejection of disturbances



$$|F(j\omega)| \approx \begin{cases} 1 & \forall\omega << \omega_c \\ |L(j\omega)| & \forall\omega >> \omega_c \end{cases}$$

The harmonic components of the disturbance outside the bandwidth are attenuated on the controlled variable.

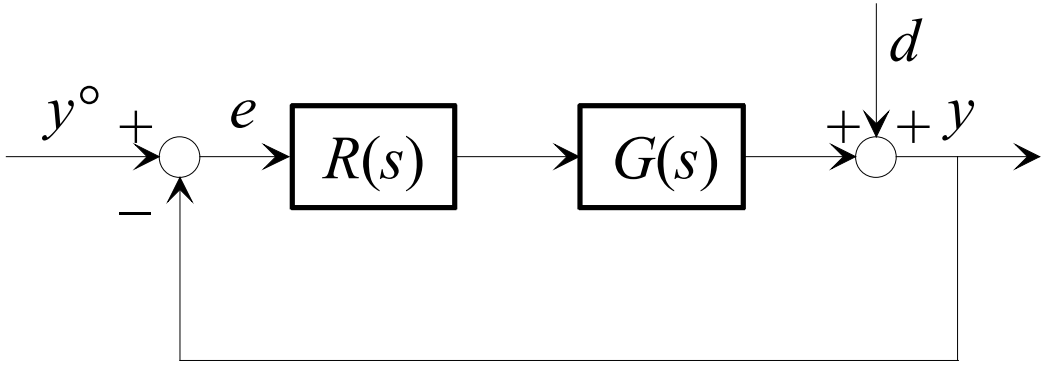
Therefore:

- the **bandwidth must not be too wide** to avoid holding the significant harmonics of the disturbance
- the smaller the magnitude of *L* outside bandwidth, the greater the attenuation

The extension of the bandwidth has to be a trade-off between different requirements.

Steady-state performance

Steady-state performance concerns the **error *e* between the reference value *y*° and the actual value *y*** of the controlled variable at steady-state.



We do not consider disturbances on the transducer

In general we achieve steady state performance either introducing an integral action in the controller or a proportional one with high gain.

Feedback: take home messages

Feedback is:

- **Good** because it allows to track the reference signal
- **Good** because it allows to reject disturbances on the process
- **Good** because it allows to get steady state performance

- **Dangerous** because it can yield unstable closed loop systems
- **Dangerous** because it brings measurement disturbances inside the loop

PID controllers

PID controllers (proportional, integral and derivative) are largely used controllers, characterized by the control law:

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)d\tau + K_D \frac{de(t)}{dt}$$

K_P: proportional gain

K_I: integral gain

K_D : derivative gain

Equivalently:

$$u(t) = K_P \left[e(t) + \frac{1}{T_I} \int_0^t e(\tau)d\tau + T_D \frac{de(t)}{dt} \right]$$

$T_I = \frac{K_P}{K_I}$ integral time

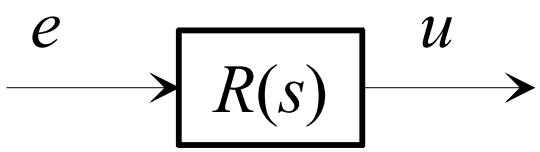
$T_D = \frac{K_D}{K_P}$ derivative time

PID's are by far the most widely used controllers in applications.

In particular, P, PD, PI and PID controllers are used.

PID: transfer function

PIDs are linear systems and as such the **transfer function** can be derived:



$$R(s) = K_P + \frac{K_I}{s} + K_D s = K_P\left(1 + \frac{1}{sT_I} + sT_D\right) = \frac{K_P}{T_I}\frac{1 + sT_I + s^2 T_I T_D}{s}$$

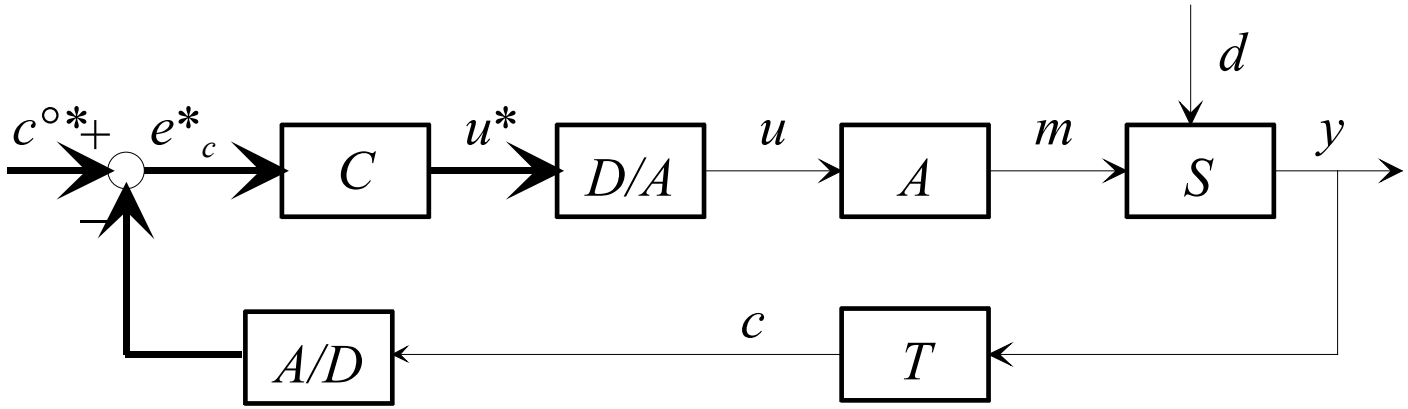
The fact that the degree of the transfer function numerator is higher than that of the denominator depends on the derivative action which is not physically feasible.

In the practical implementation of the controller, a high frequency pole should be added to the derivative action (which is, moreover, irrelevant to dynamic effects).

Digital control

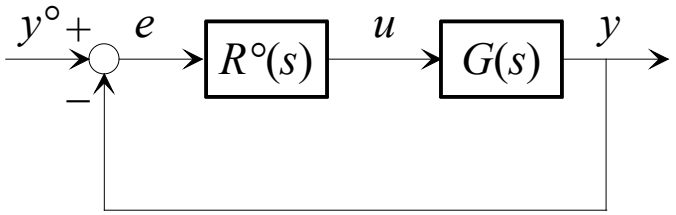
Once the transfer function of the controller has been determined, the last step is the implementation of the controller.

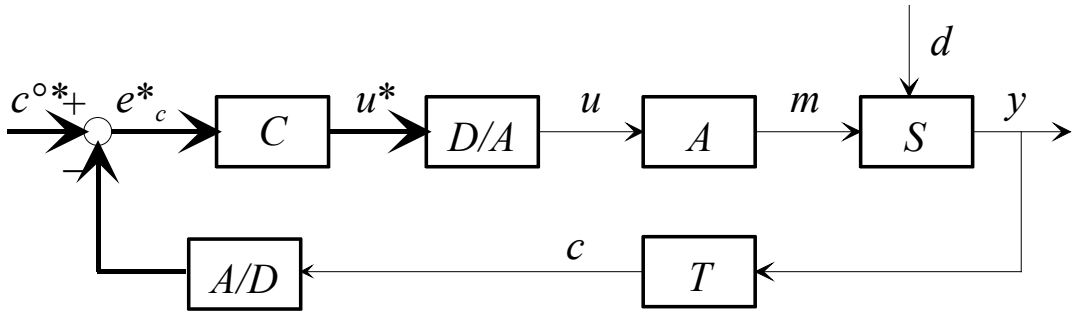
Digital control means that the controller is implemented through a **computer** (digital system) interfaced to the transducer through an Analog-to-Digital converter and to the actuator through a Digital-to-Analog converter:



Digital control

Main choices to take in a digital control system are the sampling time and the algorithm to be executed by the controller.
Basically the problem is how to evolve from an analog control system to a digital one:





- **Sampling time**: selected so as to have an adequately dense sampling of the signals. The usual criterion to select the sampling time T is to have:

$$\Omega_c = \frac{2\pi}{T} = 5 \div 10\omega_c$$

- **Algorithm**: The transfer function of the analog controller is discretized through numerical methods. The result is coded in a programming language.

Digital control

For a PI controller:

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau$$

once a sampling time *T* has been chosen, the discretized form will be:

$$u(k) = K_P e(k) + K_I x(k)$$

$$x(k) = x(k-1) + Te(k)$$

These equations will be coded and executed at each sampling time by the controller

```
input yref, y
e = yref-y;
x = x + T*e;
u = KP*e + KI*x;
```